

© λ calculus

-reducing modulo congruence relation-
-the essence and unessential in computation-

CLF reading group

October 10, 2016.

- ▶ $\mathcal{C}\mathcal{X}$ designed to come close to a diagrammatic model
- ▶ In other words - to abstract away from unessential details / syntactic bureaucracy
- ▶ Another thing - we wanted to verify that $\ast\mathcal{X}$ is suitable framework to classify all syntactically different terms which should be considered the same.

Intuitionistic logic

λx

$\lambda!x$

\Leftrightarrow

\Leftrightarrow

\Leftrightarrow

Classical logic

\mathcal{X}

$*\mathcal{X}$

$\odot\mathcal{X}$

Sequent calculus formalism

λ calculus



$*\lambda$ calculus



$\textcircled{\lambda}$ calculus

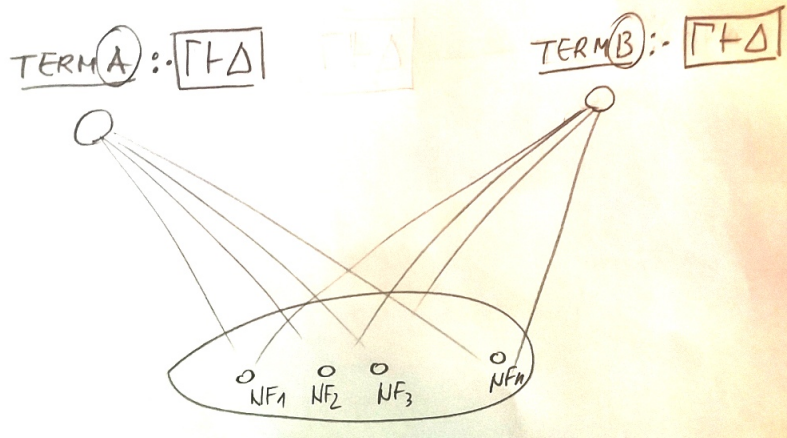
C. Urban¹ and G. Bierman

$*\lambda$ and $\textcircled{\lambda}$ have the same type system





By design all these calculi are :

- ▶ highly non-deterministic
- ▶ non-confluent
- ▶ Properties
 - ▶ linearity preservation
 - ▶ free names preservation
 - ▶ type preservation
 - ▶ termination of reduction on typed terms

Non-confluence and set of normal forms



The dagger symbol (†) : “obèle” or “obelus””

calculus	character
χ -obelix	
$*\chi$ -asterix	
$\textcircled{\chi}$ -cacophonix	
$^d\chi$ -dogmatix	

G1 sequent system for classical logic (LK)

$$\frac{}{A \vdash A} \text{ (axiom)}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \rightarrow B \vdash \Delta, \Delta'} \text{ (}\rightarrow \text{ left)}$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \text{ (}\rightarrow \text{ right)}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ (cut)}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (left weakening)}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ (right weakening)}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (left contraction)}$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ (right contraction)}$$

The terms correspond to proofs :

$$\frac{}{\langle x.\alpha \rangle :: x : A \vdash \alpha : A} \text{ (caps)}$$

$$\frac{P :: \Gamma \vdash \alpha : A, \Delta \quad Q :: \Gamma', x : B \vdash \Delta'}{P \hat{\alpha} [y] \hat{x} Q :: \Gamma, \Gamma', y : A \rightarrow B \vdash \Delta, \Delta'} \text{ (imp)} \quad \frac{P :: \Gamma, x : A \vdash \alpha : B, \Delta}{\hat{x} P \hat{\alpha} \cdot \beta :: \Gamma \vdash \beta : A \rightarrow B, \Delta} \text{ (exp)}$$

$$\frac{P :: \Gamma \vdash \alpha : A, \Delta \quad Q :: \Gamma', x : A \vdash \Delta'}{P \hat{\alpha} \dagger \hat{x} Q :: \Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ (cut)}$$

$$\frac{P :: \Gamma \vdash \Delta}{x \odot P :: \Gamma, x : A \vdash \Delta} \text{ (left eraser)}$$

$$\frac{P :: \Gamma \vdash \Delta}{P \odot \alpha :: \Gamma \vdash \alpha : A, \Delta} \text{ (right eraser)}$$

$$\frac{P :: \Gamma, y : A, z : A \vdash \Delta}{x < \hat{y} / \hat{z} [P] :: \Gamma, x : A \vdash \Delta} \text{ (left dupl.)}$$

$$\frac{P :: \Gamma \vdash \beta : A, \gamma : A, \Delta}{[P] \hat{\beta} / \hat{\gamma} > \alpha :: \Gamma \vdash \alpha : A, \Delta} \text{ (right dupl.)}$$

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The syntax

- is the same for $*\mathcal{X}$ and $\odot\mathcal{X}$

$P, Q ::= \langle x.\alpha \rangle$	<i>capsule</i>	axiom rule
$\widehat{x} P \widehat{\beta} \cdot \alpha$	<i>exporter</i>	\rightarrow R
$P \widehat{\alpha} [y] \widehat{x} Q$	<i>importer</i>	\rightarrow L
$P \widehat{\alpha} \dagger \widehat{x} Q$	<i>cut</i>	cut rule
$x \odot P$	<i>left-eraser</i>	weakening L
$P \odot \alpha$	<i>right-eraser</i>	weakening R
$x < \frac{\widehat{y}}{\widehat{z}} \langle P \rangle$	<i>left-duplicator</i>	contraction L
$[P] \widehat{\beta} \widehat{\gamma} > \alpha$	<i>right-duplicator</i>	contraction R

... Extended by **active cuts** reflecting the non-deterministic **choice** :

$P, Q ::= \dots$	
$P \widehat{\alpha} \not\bowtie \widehat{x} Q$	<i>left-active cut</i>
$P \widehat{\alpha} \searrow \widehat{x} Q$	<i>right-active cut</i>

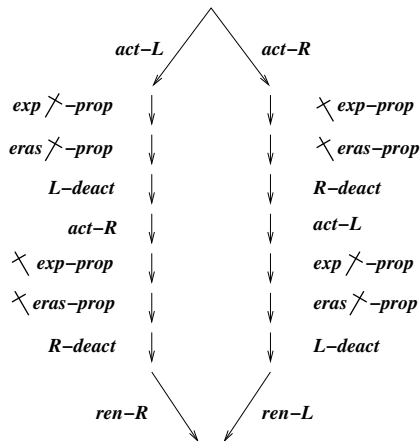
Principal name of a term

$$\begin{array}{cccc} \langle x.\alpha \rangle & \hat{x} P \hat{\beta} \cdot \alpha & P \hat{\alpha} [y] \hat{x} Q & P \hat{\alpha} \dagger \hat{x} Q \\ x \odot P & P \odot \alpha & x < \frac{\hat{y}}{\hat{z}} \langle P \rangle & [P] \frac{\hat{\beta}}{\hat{\gamma}} > \alpha \end{array}$$

- ▶ L-principal name
- ▶ S-principal name

Example reduction in $\ast\mathcal{X}$ (9 steps)

$$(\hat{x}(\langle x.\alpha \rangle \odot \beta) \hat{\beta} \cdot \gamma) \hat{\alpha} \dagger \hat{y}(\hat{z}(z \odot \langle y.\delta \rangle) \hat{\delta} \cdot \eta)$$



$$\hat{z}(\hat{x}(z \odot \langle x.\delta \rangle \odot \beta) \hat{\beta} \cdot \gamma) \hat{\delta} \cdot \eta \quad \text{OR} \quad \hat{x}(\hat{z}(z \odot \langle x.\delta \rangle \odot \beta) \hat{\delta} \cdot \eta) \hat{\beta} \cdot \gamma$$

Example of corresponding proof transformation

Remark : notice that the cut formulas (in double boxes) are introduced at the level of axioms (in oval boxes)

$$\frac{\frac{\frac{}{A \vdash \boxed{A}} (ax)}{A \vdash A, B} (wr) \quad \frac{\frac{}{\boxed{A} \vdash A} (ax)}{A, C \vdash A} (wl)}{\frac{}{\vdash \boxed{A}, A \rightarrow B} (\rightarrow R) \quad \frac{}{\boxed{A} \vdash C \rightarrow A} (\rightarrow R)}{\vdash A \rightarrow B, C \rightarrow A} (cut)$$

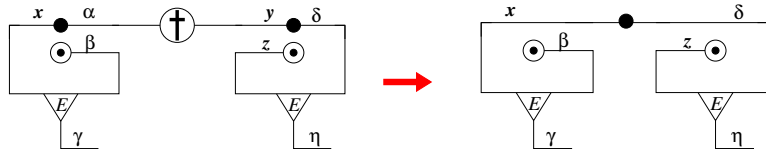
This proof normalizes (also in 9 steps) to either

$$\frac{\frac{\frac{\frac{}{A \vdash A} (ax)}{A, C \vdash A, B} (wr, wl)}{C \vdash A \rightarrow B, A} (\rightarrow R^1)}{\vdash A \rightarrow B, C \rightarrow A} (\rightarrow R^2)$$

or

$$\frac{\frac{\frac{\frac{}{A \vdash A} (ax)}{A, C \vdash A, B} (wr, wl)}{A \vdash B, C \rightarrow A} (\rightarrow R^2)}{\vdash A \rightarrow B, C \rightarrow A} (\rightarrow R^1)$$

In 2d - a single reduction step :



Revisit the example in the framework of $\odot \mathcal{X}$

The computation should², ideally, go as follows :

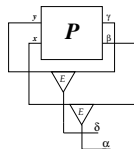
$$\begin{aligned} & (\hat{x}(\langle x.\alpha \rangle \odot \beta) \hat{\beta} \cdot \gamma) \hat{\alpha} \dagger \hat{y}(\hat{z}(z \odot \langle y.\delta \rangle) \hat{\delta} \cdot \eta) \\ \equiv & \hat{x}(\hat{z}(z \odot (\langle x.\alpha \rangle \hat{\alpha} \dagger \hat{y}\langle y.\delta \rangle)) \odot \beta) \hat{\delta} \cdot \eta) \hat{\beta} \cdot \gamma \\ \rightarrow & \hat{x}(\hat{z}(z \odot \langle x.\delta \rangle) \odot \beta) \hat{\delta} \cdot \eta) \hat{\beta} \cdot \gamma \end{aligned}$$

2. We later prove that this is indeed always possible: 

- ▶ few reduction rules
- ▶ many congruence rules
- ▶ reducing is reducing modulo congruence relation

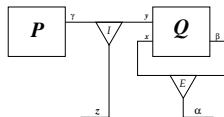
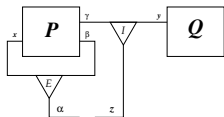
Congruence rules (relation \equiv)

1. exp - exp



$$\widehat{y}(\widehat{x} P \widehat{\beta} \cdot \alpha) \widehat{\gamma} \cdot \delta \equiv \widehat{x}(\widehat{y} P \widehat{\gamma} \cdot \delta) \widehat{\beta} \cdot \alpha$$

2. exp - imp



$$\widehat{x}(P \widehat{\gamma} [z] \widehat{y} Q) \widehat{\beta} \cdot \alpha \equiv (\widehat{x} P \widehat{\beta} \cdot \alpha) \widehat{\gamma} [z] \widehat{y} Q$$

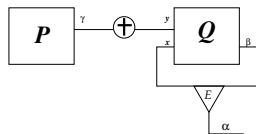
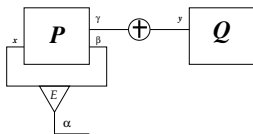
$$\widehat{x}(P \widehat{\gamma} [z] \widehat{y} Q) \widehat{\beta} \cdot \alpha \equiv P \widehat{\gamma} [z] \widehat{y}(\widehat{x} Q \widehat{\beta} \cdot \alpha)$$

with $x, \beta \in N(P)$

with $x, \beta \in N(Q)$

Congruence rules

3. exporter - cut



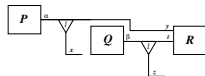
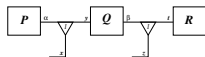
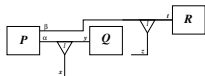
$$\widehat{x}(P\widehat{\gamma} \dagger \widehat{y}Q)\widehat{\beta} \cdot \alpha \equiv (\widehat{x}P\widehat{\beta} \cdot \alpha)\widehat{\gamma} \dagger \widehat{y}Q$$

with $x, \beta \in P$

$$\widehat{x}(P\widehat{\gamma} \dagger \widehat{y}Q)\widehat{\beta} \cdot \alpha \equiv P\widehat{\gamma} \dagger \widehat{y}(\widehat{x}Q\widehat{\beta} \cdot \alpha)$$

with $x, \beta \in Q$

4. importer - importer



$$(P\widehat{\alpha} [x] \widehat{y}Q)\widehat{\beta} [z] \widehat{t}R \equiv (P\widehat{\beta} [z] \widehat{t}R)\widehat{\alpha} [x] \widehat{y}Q$$

with $\alpha, \beta \in P$

$$(P\widehat{\alpha} [x] \widehat{y}Q)\widehat{\beta} [z] \widehat{t}R \equiv P\widehat{\alpha} [x] \widehat{y}(Q\widehat{\beta} [z] \widehat{t}R)$$

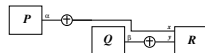
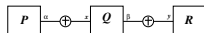
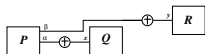
with $y, \beta \in Q$

$$P\widehat{\alpha} [x] \widehat{y}(Q\widehat{\beta} [z] \widehat{t}R) \equiv Q\widehat{\beta} [z] \widehat{t}(P\widehat{\alpha} [x] \widehat{y}R)$$

with $y, t \in R$

Congruence rules

5. cut - cut



$$(P\hat{\alpha} \dagger \hat{x}Q)\hat{\beta} \dagger \hat{y}R \equiv (P\hat{\beta} \dagger \hat{y}R)\hat{\alpha} \dagger \hat{x}Q$$

$$(P\hat{\alpha} \dagger \hat{x}Q)\hat{\beta} \dagger \hat{y}R \equiv P\hat{\alpha} \dagger \hat{x}(Q\hat{\beta} \dagger \hat{y}R)$$

$$P\hat{\alpha} \dagger \hat{x}(Q\hat{\beta} \dagger \hat{y}R) \equiv Q\hat{\beta} \dagger \hat{y}(P\hat{\alpha} \dagger \hat{x}R)$$

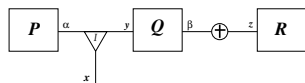
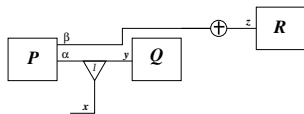
with $\alpha, \beta \in N(P)$

with $x, \beta \in N(Q)$

with $x, y \in N(R)$

Congruence rules

6. cut - importer

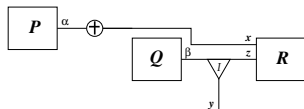
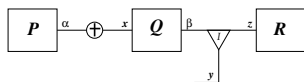


$$(P \hat{\alpha} [x] \hat{y} Q) \hat{\beta} \dagger \hat{z} R \equiv (P \hat{\beta} \dagger \hat{z} R) \hat{\alpha} [x] \hat{y} Q$$

with $\alpha, \beta \in N(P)$

$$(P \hat{\alpha} [x] \hat{y} Q) \hat{\beta} \dagger \hat{z} R \equiv P \hat{\alpha} [x] \hat{y} (Q \hat{\beta} \dagger \hat{z} R)$$

with $y, \beta \in N(Q)$



$$P \hat{\alpha} \dagger \hat{x} (Q \hat{\beta} [y] \hat{z} R) \equiv (P \hat{\alpha} \dagger \hat{x} Q) \hat{\beta} [y] \hat{z} R$$

with $x, \beta \in N(Q)$

$$P \hat{\alpha} \dagger \hat{x} (Q \hat{\beta} [y] \hat{z} R) \equiv Q \hat{\beta} [y] \hat{z} (P \hat{\alpha} \dagger \hat{x} R)$$

with $x, z \in N(R)$

The rest of the congruence rules are omitted [here](#).

Congruence relation, \equiv

- ▶ the relation \equiv induces congruence classes on terms
- ▶ $cl(P)$ denotes the congruence class of a term P with respect to a relation \equiv
- ▶ each congruence class has finitely many terms

Restructuring terms

Take for example an arbitrary λ -term of the form :

$$P\hat{\alpha} \dagger \hat{x}Q$$

Here, the name α might not be directly accessible. Furthermore, both α and x may occur deep inside their terms. We prove that it is possible to transform the above term - using only congruence rules - to the form

$$C\{P^\alpha\hat{\alpha} \dagger \hat{x}Q^x\}$$

where C is a context, and α and x are principal names of $P^\alpha \preceq P$ and $Q^x \preceq Q$, respectively.

In other words we can always pick at least one representative of a congruence class $cl(P\hat{\alpha} \dagger \hat{x}Q)$, and enable to continue the reduction process. RECALL : \preceq stands for a subterm relation. $cl(P)$ denotes the congruence class of a term P with respect to \equiv .

Restructuring - proof sketch

Lemma - left-side

For every term of the form $P\hat{\alpha} \dagger \hat{x}Q$, there exists a context C and a term P^α , where α is a principal name for P^α and $P^\alpha \preceq P$, such that

$$P\hat{\alpha} \dagger \hat{x}Q \equiv C\{P^\alpha\hat{\alpha} \dagger \hat{x}Q\}$$

Proof : By induction on the structure of P and case analysis.

Lema - right-side

For every term of the form $P\hat{\alpha} \dagger \hat{x}Q$, there exists a context C and a term Q^x , whose principal name is x and $Q^x \preceq Q$, such that

$$P\hat{\alpha} \dagger \hat{x}Q \equiv C\{P\hat{\alpha} \dagger \hat{x}Q^x\}$$

Proof : By induction on the structure of Q and case analysis.

Restructuring - proof sketch

Theorem

For every term of the form $P\hat{\alpha} \dagger \hat{x}Q$ there exists a context C and terms P^α and Q^x whose principal names are α, x respectively. and $P^\alpha \preceq P, Q^x \preceq Q$, such that

$$P\hat{\alpha} \dagger \hat{x}Q \equiv C\{P^\alpha\hat{\alpha} \dagger \hat{x}Q^x\}$$

Proof : By using the two previous lemmas, we can construct the proof in two symmetric ways. One is

$$\begin{aligned} P\hat{\alpha} \dagger \hat{x}Q &\stackrel{\text{lemma 1}}{\equiv} C_1\{P^\alpha\hat{\alpha} \dagger \hat{x}Q\} \\ &\stackrel{\text{lemma 2}}{\equiv} C_1\{C_2\{P^\alpha\hat{\alpha} \dagger \hat{x}Q^x\}\} \\ &\triangleq C'\{P^\alpha\hat{\alpha} \dagger \hat{x}Q^x\}, \quad \text{with } C'\{\} = C_1\{C_2\{\}\} \end{aligned}$$

Reduction rules

duality between activation / deactivation

Activation

$(act-L) : P\hat{\alpha} \dagger \hat{x}Q \rightarrow P\hat{\alpha} \not\! \times \hat{x}Q, \alpha \text{ is S-principal for } P$

$(act-R) : P\hat{\alpha} \dagger \hat{x}Q \rightarrow P\hat{\alpha} \not\! \backslash \hat{x}Q, x \text{ is S-principal for } Q$

Deactivation

$(\not\! \times\text{-deact}) : P\hat{\alpha} \not\! \times \hat{x}Q \rightarrow P\hat{\alpha} \dagger \hat{x}Q, \text{ if } \alpha \text{ is L-principal for } P$

$(\not\! \backslash\text{-deact}) : P\hat{\alpha} \not\! \backslash \hat{x}Q \rightarrow P\hat{\alpha} \dagger \hat{x}Q, \text{ if } x \text{ is L-principal for } Q$

Reduction rules

symmetry between left / right rules

Structural rules

Left :

$$(\text{\textbackslash-eras}) : (P \odot \alpha) \hat{\alpha} \text{\textbackslash} \hat{x} Q \rightarrow \mathcal{I}^Q \odot P \odot \mathcal{O}^Q$$

$$(\text{\textbackslash-dupl}) : ([P]_{\alpha_2}^{\alpha_1} > \alpha) \hat{\alpha} \text{\textbackslash} \hat{x} Q \rightarrow P \langle\langle \hat{\alpha}_1 \hat{\alpha}_2 \text{\textbackslash} \hat{x} Q \rangle\rangle$$

Right :

$$(\text{\textbackslash-eras}) : P \hat{\alpha} \text{\textbackslash} \hat{x} (x \odot Q) \rightarrow \mathcal{I}^P \odot Q \odot \mathcal{O}^P$$

$$(\text{\textbackslash-dupl}) : P \hat{\alpha} \text{\textbackslash} \hat{x} (x < \frac{\hat{x}_1}{\hat{x}_2} \langle Q \rangle) \rightarrow \langle\langle P \hat{\alpha} \text{\textbackslash} \hat{x}_1 \hat{x}_2 \rangle\rangle Q$$

Reduction rules

symmetry between left / right rules

Logical rules

$$(ren-L) \quad : \quad \langle y.\alpha \rangle \hat{\alpha} \dagger \hat{x}Q \quad \rightarrow \quad Q\{y/x\}$$

$$(ren-R) \quad : \quad P\hat{\alpha} \dagger \hat{x}\langle x.\beta \rangle \quad \rightarrow \quad P\{\beta/\alpha\}$$

$$(ei-insert) \quad : \quad (\hat{y}P\hat{\beta}.\alpha)\hat{\alpha} \dagger \hat{x}(Q\hat{\gamma} [x] \hat{z}R) \quad \rightarrow \quad \begin{cases} (Q\hat{\gamma} \dagger \hat{y}P)\hat{\beta} \dagger \hat{z}R \\ Q\hat{\gamma} \dagger \hat{y}(P\hat{\beta} \dagger \hat{z}R) \end{cases}$$

Theorem - preservation of free names, by \equiv and \rightarrow

If $P \equiv Q$ then $N(P) = N(Q)$

If $P \rightarrow Q$ then $N(P) = N(Q)$

Theorem - preservation of linearity, by \equiv and \rightarrow

If P is linear and $P \equiv Q$ then Q is linear

If P is linear and $P \rightarrow Q$ then Q is linear

Proof : By treating carefully each rule.

Operational properties

Theorem - preservation of types, by \equiv and \rightarrow

Let S be an $\textcircled{C}\mathcal{X}$ -term and Γ, Δ contexts. Then the following holds :

If $S :: \Gamma \vdash \Delta$ and $S \equiv S'$, then $S' :: \Gamma \vdash \Delta$

If $S :: \Gamma \vdash \Delta$ and $S \rightarrow S'$, then $S' :: \Gamma \vdash \Delta$

Proof : By checking that this holds for all equations and reduction rules.

Theorem - strong normalisation³

The reduction system of $^*\mathcal{X}$ is strongly normalising on simply-typed terms.

Proof : Preservation of strong normalisation, based on SN of \mathcal{X} and $^*\mathcal{X}$.

3. Not yet formally proved

- ▶ You may check a draft article at this URL :
<https://sites.google.com/site/dragisazunic/Home/research/CX.pdf>
- ▶ Also, my PhD thesis available at :
<http://tel.archives-ouvertes.fr/tel-00265549>

Thank you