$^{©}\mathcal{X}$ calculus

-reducing modulo congruence relation-
-the essence and unessential in computation-

CLF reading group

October 10, 2016.

Intro

- $^{\copyright}\mathcal{X}$ designed to come close to a diagrammatic model
- In other words - to abstract away from unessential details / syntactic bureaucrac
- Another thing - we wanted to verify that $^{*}\mathcal{X}$ is suitable framework to classify all syntactically different terms which should be considered the same.

Perspective

Intuitionistic logic		Classical logic
λx	$\Leftrightarrow$	$\mathcal{X}$
λlxr	$\Leftrightarrow$	$^{*}\mathcal{X}$
	$\Leftrightarrow$	©$\mathcal{X}$

Sequent calculus formalism

Perspective

$\mathcal{X}$ calculus

$\downarrow$

$^*\mathcal{X}$ calculus

$\downarrow$

$^{©}\mathcal{X}$ calculus

C. Urban [1] and G. Bierman

$^*\mathcal{X}$ and $^{©}\mathcal{X}$ have the same type system

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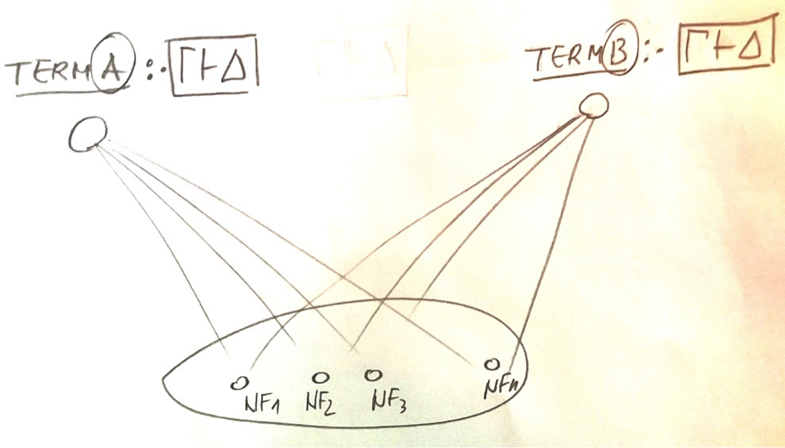
By design all these calculi are :

- highly non-deterministic
- non-confluent
- Properties
  - linearity preservation
  - free names preservation
  - type preservation
  - termination of reduction on typed terms

1. ... in his PhD, University of Cambridge, 2000.

Non-confluence and set of normal forms



The dagger symbol (†) : “obèle” or “obelus””

calculus	character
$\mathcal{X}$ -obelix	
$^*\mathcal{X}$ -asterix	
$^{\copyright}\mathcal{X}$ -cacofonix	
$^d\mathcal{X}$ -dogmatix	

G1 sequent system for classical logic (LK)

$$\frac{}{A \vdash A} \text{ (axiom)}$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \to B \vdash \Delta, \Delta'} \text{ ($\to$ left)} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \text{ ($\to$ right)}$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ (cut)}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (left weakening)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ (right weakening)}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (left contraction)} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ (right contraction)}$$

The terms correspond to proofs :

$$\frac{}{\langle x.\alpha\rangle :\cdot\ x : A \vdash \alpha : A}\ \text{(caps)}$$

$$\frac{P :\cdot\ \Gamma \vdash \alpha : A, \Delta \qquad Q :\cdot\ \Gamma', x : B \vdash \Delta'}{P\widehat{\alpha}\ [y]\ \widehat{x}Q :\cdot\ \Gamma, \Gamma', y : A \to B \vdash \Delta, \Delta'}\ \text{(imp)} \qquad \frac{P :\cdot\ \Gamma, x : A \vdash \alpha : B, \Delta}{\widehat{x}\, P\, \widehat{\alpha} \cdot \beta :\cdot\ \Gamma \vdash \beta : A \to B, \Delta}\ \text{(exp)}$$

$$\frac{P :\cdot\ \Gamma \vdash \alpha : A, \Delta \qquad Q :\cdot\ \Gamma', x : A \vdash \Delta'}{P\widehat{\alpha}\ \dagger\ \widehat{x}Q :\cdot\ \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{(cut)}$$

$$\frac{P :\cdot\ \Gamma \vdash \Delta}{x \odot P :\cdot\ \Gamma, x : A \vdash \Delta}\ \text{(left eraser)} \qquad \frac{P :\cdot\ \Gamma \vdash \Delta}{P \odot \alpha :\cdot\ \Gamma \vdash \alpha : A, \Delta}\ \text{(right eraser)}$$

$$\frac{P :\cdot\ \Gamma, y : A, z : A \vdash \Delta}{x {<}\tfrac{\widehat{y}}{\widehat{z}}\langle P] :\cdot\ \Gamma, x : A \vdash \Delta}\ \text{(left dupl.)} \qquad \frac{P :\cdot\ \Gamma \vdash \beta : A, \gamma : A, \Delta}{[P\rangle\tfrac{\widehat{\beta}}{\widehat{\gamma}}{>}\alpha :\cdot\ \Gamma \vdash \alpha : A, \Delta}\ \text{(right dupl.)}$$

The terms correspond to proofs :

$$\frac{}{\langle x.\alpha\rangle \;\because\; x : A \vdash \alpha : A}\;(\text{caps})$$

$$\frac{P \;\because\; \Gamma \vdash \alpha : A, \Delta \qquad Q \;\because\; \Gamma', x : B \vdash \Delta'}{P\widehat{\alpha}\;[y]\;\widehat{x}Q \;\because\; \Gamma, \Gamma', y : A \rightarrow B \vdash \Delta, \Delta'}\;(\text{imp}) \qquad \frac{P \;\because\; \Gamma, x : A \vdash \alpha : B, \Delta}{\widehat{x}\,P\,\widehat{\alpha}\cdot\beta \;\because\; \Gamma \vdash \beta : A \rightarrow B, \Delta}\;(\text{exp})$$

$$\frac{P \;\because\; \Gamma \vdash \alpha : A, \Delta \qquad Q \;\because\; \Gamma', x : A \vdash \Delta'}{P\widehat{\alpha}\;\dagger\;\widehat{x}Q \;\because\; \Gamma, \Gamma' \vdash \Delta, \Delta'}\;(\text{cut})$$

$$\frac{P \;\because\; \Gamma \vdash \Delta}{x \odot P \;\because\; \Gamma, x : A \vdash \Delta}\;(\text{left eraser}) \qquad \frac{P \;\because\; \Gamma \vdash \Delta}{P \odot \alpha \;\because\; \Gamma \vdash \alpha : A, \Delta}\;(\text{right eraser})$$

$$\frac{P \;\because\; \Gamma, y : A, z : A \vdash \Delta}{x{<}^{\widehat{y}}_{\widehat{z}}\langle P] \;\because\; \Gamma, x : A \vdash \Delta}\;(\text{left dupl.}) \qquad \frac{P \;\because\; \Gamma \vdash \beta : A, \gamma : A, \Delta}{[P\rangle^{\widehat{\beta}}_{\widehat{\gamma}}{>}\alpha \;\because\; \Gamma \vdash \alpha : A, \Delta}\;(\text{right dupl.})$$

The terms correspond to proofs :

$$\frac{}{\langle x.\alpha\rangle :\cdot\ x : A \vdash \alpha : A}\ \text{(caps)}$$

$$\frac{P :\cdot\ \Gamma \vdash \alpha : A, \Delta \qquad Q :\cdot\ \Gamma', x : B \vdash \Delta'}{P\widehat{\alpha}\ [y]\ \widehat{x}Q :\cdot\ \Gamma, \Gamma', y : A \to B \vdash \Delta, \Delta'}\ \text{(imp)} \qquad \frac{P :\cdot\ \Gamma, x : A \vdash \alpha : B, \Delta}{\widehat{x}\, P\, \widehat{\alpha} \cdot \beta :\cdot\ \Gamma \vdash \beta : A \to B, \Delta}\ \text{(exp)}$$

$$\frac{P :\cdot\ \Gamma \vdash \alpha : A, \Delta \qquad Q :\cdot\ \Gamma', x : A \vdash \Delta'}{P\widehat{\alpha}\ \dagger\ \widehat{x}Q :\cdot\ \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{(cut)}$$

$$\frac{P :\cdot\ \Gamma \vdash \Delta}{x \odot P :\cdot\ \Gamma, x : A \vdash \Delta}\ \text{(left eraser)} \qquad \frac{P :\cdot\ \Gamma \vdash \Delta}{P \odot \alpha :\cdot\ \Gamma \vdash \alpha : A, \Delta}\ \text{(right eraser)}$$

$$\frac{P :\cdot\ \Gamma, y : A, z : A \vdash \Delta}{x{<}\tfrac{\widehat{y}}{\widehat{z}}\langle P] :\cdot\ \Gamma, x : A \vdash \Delta}\ \text{(left dupl.)} \qquad \frac{P :\cdot\ \Gamma \vdash \beta : A, \gamma : A, \Delta}{[P\rangle\tfrac{\widehat{\beta}}{\widehat{\gamma}}{>}\alpha :\cdot\ \Gamma \vdash \alpha : A, \Delta}\ \text{(right dupl.)}$$

The terms correspond to proofs :

$$\frac{}{\langle x.\alpha\rangle \;:\!\cdot\; x : A \vdash \alpha : A}\ \text{(caps)}$$

$$\frac{P \;:\!\cdot\; \Gamma \vdash \alpha : A, \Delta \qquad Q \;:\!\cdot\; \Gamma', x : B \vdash \Delta'}{P\widehat{\alpha}\ [y]\ \widehat{x}Q \;:\!\cdot\; \Gamma, \Gamma', y : A \to B \vdash \Delta, \Delta'}\ \text{(imp)} \qquad \frac{P \;:\!\cdot\; \Gamma, x : A \vdash \alpha : B, \Delta}{\widehat{x}\, P\, \widehat{\alpha} \cdot \beta \;:\!\cdot\; \Gamma \vdash \beta : A \to B, \Delta}\ \text{(exp)}$$

$$\frac{P \;:\!\cdot\; \Gamma \vdash \alpha : A, \Delta \qquad Q \;:\!\cdot\; \Gamma', x : A \vdash \Delta'}{P\widehat{\alpha}\ \dagger\ \widehat{x}Q \;:\!\cdot\; \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{(cut)}$$

$$\frac{P \;:\!\cdot\; \Gamma \vdash \Delta}{x \odot P \;:\!\cdot\; \Gamma, x : A \vdash \Delta}\ \text{(left eraser)} \qquad \frac{P \;:\!\cdot\; \Gamma \vdash \Delta}{P \odot \alpha \;:\!\cdot\; \Gamma \vdash \alpha : A, \Delta}\ \text{(right eraser)}$$

$$\frac{P \;:\!\cdot\; \Gamma, y : A, z : A \vdash \Delta}{x{<}^{\widehat{y}}_{\widehat{z}}\langle P] \;:\!\cdot\; \Gamma, x : A \vdash \Delta}\ \text{(left dupl.)} \qquad \frac{P \;:\!\cdot\; \Gamma \vdash \beta : A, \gamma : A, \Delta}{[P\rangle^{\widehat{\beta}}_{\widehat{\gamma}}{>}\alpha \;:\!\cdot\; \Gamma \vdash \alpha : A, \Delta}\ \text{(right dupl.)}$$

The terms correspond to proofs :

$$\frac{}{\langle x.\alpha\rangle :\cdot\ x:A\vdash \alpha:A}\ \text{(caps)}$$

$$\frac{P :\cdot\ \Gamma\vdash \alpha:A,\Delta \qquad Q :\cdot\ \Gamma',x:B\vdash\Delta'}{P\widehat{\alpha}\ [y]\ \widehat{x}Q :\cdot\ \Gamma,\Gamma',y:A\to B\vdash\Delta,\Delta'}\ \text{(imp)} \qquad \frac{P :\cdot\ \Gamma,x:A\vdash\alpha:B,\Delta}{\widehat{x}\,P\,\widehat{\alpha}\cdot\beta :\cdot\ \Gamma\vdash\beta:A\to B,\Delta}\ \text{(exp)}$$

$$\frac{P :\cdot\ \Gamma\vdash\alpha:A,\Delta \qquad Q :\cdot\ \Gamma',x:A\vdash\Delta'}{P\widehat{\alpha}\ \dagger\ \widehat{x}Q :\cdot\ \Gamma,\Gamma'\vdash\Delta,\Delta'}\ \text{(cut)}$$

$$\frac{P :\cdot\ \Gamma\vdash\Delta}{x\odot P :\cdot\ \Gamma,x:A\vdash\Delta}\ \text{(left eraser)} \qquad \frac{P :\cdot\ \Gamma\vdash\Delta}{P\odot\alpha :\cdot\ \Gamma\vdash\alpha:A,\Delta}\ \text{(right eraser)}$$

$$\frac{P :\cdot\ \Gamma,y:A,z:A\vdash\Delta}{x{<}^{\widehat{y}}_{\widehat{z}}\langle P] :\cdot\ \Gamma,x:A\vdash\Delta}\ \text{(left dupl.)} \qquad \frac{P :\cdot\ \Gamma\vdash\beta:A,\gamma:A,\Delta}{[P\rangle^{\widehat{\beta}}_{\widehat{\gamma}}{>}\alpha :\cdot\ \Gamma\vdash\alpha:A,\Delta}\ \text{(right dupl.)}$$

The syntax

- is the same for ${}^{*}\mathcal{X}$ and ${}^{©}\mathcal{X}$

$P,\ Q$	$::=$	$\langle x.\alpha \rangle$	*capsule*	axiom rule
	$\mid$	$\widehat{x}\, P\, \widehat{\beta} \cdot \alpha$	*exporter*	$\rightarrow$ R
	$\mid$	$P \widehat{\alpha}\ [y]\ \widehat{x} Q$	*importer*	$\rightarrow$ L
	$\mid$	$P \widehat{\alpha}\ \dagger\ \widehat{x} Q$	*cut*	cut rule
	$\mid$	$x \odot P$	*left-eraser*	weakening L
	$\mid$	$P \odot \alpha$	*right-eraser*	weakening R
	$\mid$	$x {<}^{\widehat{y}}_{\widehat{z}} \langle P]$	*left-duplicator*	contraction L
	$\mid$	$[P\rangle^{\widehat{\beta}}_{\widehat{\gamma}}{>}\alpha$	*right-duplicator*	contraction R

... Extended by active cuts reflecting the non-deterministic choice :

$$
\begin{array}{lcll}
P, Q & ::= & \cdots & \\
 & \mid & P\widehat{\alpha} \nearrow \widehat{x} Q & \textit{left-active cut} \\
 & \mid & P\widehat{\alpha} \nwarrow \widehat{x} Q & \textit{right-active cut}
\end{array}
$$

Principal name of a term

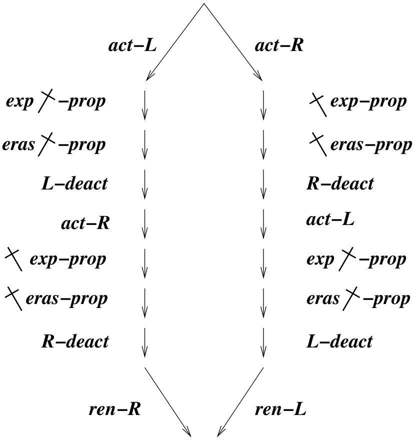
$$\langle x.\alpha \rangle \qquad \widehat{x}\, P\, \widehat{\beta} \cdot \alpha \qquad P\widehat{\alpha}\ [y]\ \widehat{x}Q \qquad P\widehat{\alpha}\ \dagger\ \widehat{x}Q$$

$$x \odot P \qquad P \odot \alpha \qquad x {<}^{\widehat{y}}_{\widehat{z}} \langle P] \qquad [P\rangle^{\widehat{\beta}}_{\widehat{\gamma}} {>} \alpha$$

- L–principal name
- S–principal name

Example reduction in ${}^{*}\mathcal{X}$ (9 steps)

$$(\widehat{x}(\langle x.\alpha\rangle \odot \beta)\widehat{\beta}\cdot\gamma)\widehat{\alpha} \dagger \widehat{y}(\widehat{z}(z\odot\langle y.\delta\rangle)\widehat{\delta}\cdot\eta)$$



$$\widehat{z}(\widehat{x}(z\odot \langle x.\delta\rangle \odot\beta)\widehat{\beta}\cdot\gamma)\widehat{\delta}\cdot\eta \quad \text{OR} \quad \widehat{x}(\widehat{z}(z\odot\langle x.\delta\rangle\odot\beta)\widehat{\delta}\cdot\eta)\widehat{\beta}\cdot\gamma$$

Example of corresponding proof transformation

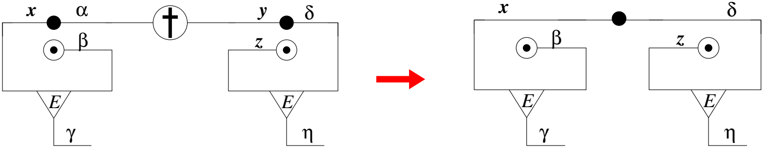
Remark : notice that the cut formulas (in double boxes) are introduced at the level of axioms (in oval boxes)

$$
\dfrac{
\dfrac{\dfrac{\dfrac{}{A \vdash \boxed{A}}\ (ax)}{A \vdash A, B}\ (wr)}{\vdash \boxed{\boxed{A}}, A \to B}\ (\to R)
\qquad
\dfrac{\dfrac{\dfrac{}{\boxed{A} \vdash A}\ (ax)}{A, C \vdash A}\ (wl)}{\boxed{\boxed{A}} \vdash C \to A}\ (\to R)
}{\vdash A \to B,\ C \to A}\ (cut)
$$

This proof normalizes (<u>also in 9 steps</u>) to either

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ (ax)}{A, C \vdash A, B}\ (wr, wl)}{C \vdash A \to B,\ A}\ (\to R^1)}{\vdash A \to B,\ C \to A}\ (\to R^2)
\qquad \text{or} \qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ (ax)}{A, C \vdash A, B}\ (wr, wl)}{A \vdash B,\ C \to A}\ (\to R^2)}{\vdash A \to B,\ C \to A}\ (\to R^1)
$$

In 2d - a single reduction step :



Revisit the example in the framework of ${}^{\copyright}\mathcal{X}$

The computation should[2], ideally, go as follows :

$$
\begin{array}{rl}
& (\widehat{x}\,(\underline{\langle x.\alpha\rangle}\odot\beta)\,\widehat{\beta}\cdot\gamma)\widehat{\alpha}\;\dagger\;\widehat{y}(\widehat{z}\,(z\odot\underline{\langle y.\delta\rangle})\,\widehat{\delta}\cdot\eta) \\
\equiv & \widehat{x}\,(\widehat{z}\,(z\odot\;(\underline{\langle x.\alpha\rangle\widehat{\alpha}\;\dagger\;\widehat{y}\langle y.\delta\rangle})\;\odot\beta)\,\widehat{\delta}\cdot\eta)\,\widehat{\beta}\cdot\gamma \\
\rightarrow & \widehat{x}\,(\widehat{z}\,(z\odot\;\underline{\langle x.\delta\rangle}\;\odot\beta)\,\widehat{\delta}\cdot\eta)\,\widehat{\beta}\cdot\gamma
\end{array}
$$

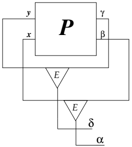
2. We later prove that this is indeed always possible.

So... in ©$\mathcal{X}$

- few reduction rules
- many congruence rules
- reducing is reducing modulo congruence relation

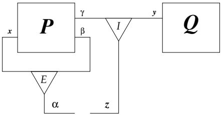
Congruence rules (relation $\equiv$)

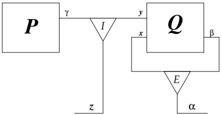
1. exp - exp



$$\widehat{y}\,(\widehat{x}\,P\,\widehat{\beta}\cdot\alpha)\,\widehat{\gamma}\cdot\delta \equiv \widehat{x}\,(\widehat{y}\,P\,\widehat{\gamma}\cdot\delta)\,\widehat{\beta}\cdot\alpha$$

2. exp - imp



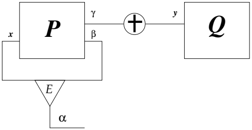


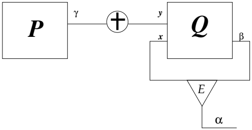
$$\widehat{x}\,(P\,\widehat{\gamma}\;[z]\;\widehat{y}\;Q)\,\widehat{\beta}\cdot\alpha \;\equiv\; (\widehat{x}\,P\,\widehat{\beta}\cdot\alpha)\,\widehat{\gamma}\;[z]\;\widehat{y}\;Q \qquad \text{with } x,\beta\in N(P)$$

$$\widehat{x}\,(P\,\widehat{\gamma}\;[z]\;\widehat{y}\;Q)\,\widehat{\beta}\cdot\alpha \;\equiv\; P\,\widehat{\gamma}\;[z]\;\widehat{y}\,(\widehat{x}\,Q\,\widehat{\beta}\cdot\alpha) \qquad \text{with } x,\beta\in N(Q)$$

Congruence rules

3. exporter - cut

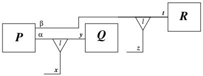


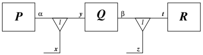


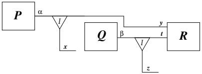
$$\widehat{x}\,(P\widehat{\gamma}\ \dagger\ \widehat{y}Q)\,\widehat{\beta}\cdot\alpha \;\equiv\; (\widehat{x}\,P\,\widehat{\beta}\cdot\alpha)\widehat{\gamma}\ \dagger\ \widehat{y}Q \qquad \text{with } x,\beta \in P$$

$$\widehat{x}\,(P\widehat{\gamma}\ \dagger\ \widehat{y}Q)\,\widehat{\beta}\cdot\alpha \;\equiv\; P\widehat{\gamma}\ \dagger\ \widehat{y}(\widehat{x}\,Q\,\widehat{\beta}\cdot\alpha) \qquad \text{with } x,\beta \in Q$$

4. importer - importer







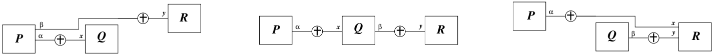
$$(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\,\widehat{\beta}\,[z]\,\widehat{t}\,R \;\equiv\; (P\,\widehat{\beta}\,[z]\,\widehat{t}\,R)\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q \qquad \text{with } \alpha,\beta \in P$$

$$(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\,\widehat{\beta}\,[z]\,\widehat{t}\,R \;\equiv\; P\,\widehat{\alpha}\,[x]\,\widehat{y}\,(Q\,\widehat{\beta}\,[z]\,\widehat{t}\,R) \qquad \text{with } y,\beta \in Q$$

$$P\,\widehat{\alpha}\,[x]\,\widehat{y}\,(Q\,\widehat{\beta}\,[z]\,\widehat{t}\,R) \;\equiv\; Q\,\widehat{\beta}\,[z]\,\widehat{t}\,(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,R) \qquad \text{with } y,t \in R$$

Congruence rules

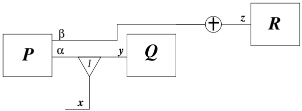
5. cut - cut

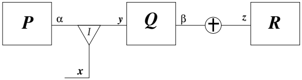


$$
\begin{aligned}
(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}R &\equiv (P\widehat{\beta} \dagger \widehat{y}R)\widehat{\alpha} \dagger \widehat{x}Q && \text{with } \alpha, \beta \in N(P) \\
(P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta} \dagger \widehat{y}R &\equiv P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R) && \text{with } x, \beta \in N(Q) \\
P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta} \dagger \widehat{y}R) &\equiv Q\widehat{\beta} \dagger \widehat{y}(P\widehat{\alpha} \dagger \widehat{x}R) && \text{with } x, y \in N(R)
\end{aligned}
$$

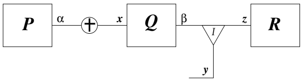
Congruence rules

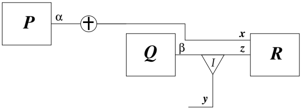
6. cut - importer





$$
\begin{array}{rcll}
(P\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\widehat{\beta} \dagger \widehat{z}R & \equiv & (P\widehat{\beta} \dagger \widehat{z}R)\widehat{\alpha}\,[x]\,\widehat{y}\,Q & \text{with } \alpha, \beta \in N(P) \\
(P\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\widehat{\beta} \dagger \widehat{z}R & \equiv & P\widehat{\alpha}\,[x]\,\widehat{y}\,(Q\widehat{\beta} \dagger \widehat{z}R) & \text{with } y, \beta \in N(Q)
\end{array}
$$





$$
\begin{array}{rcll}
P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[y]\,\widehat{z}\,R) & \equiv & (P\widehat{\alpha} \dagger \widehat{x}Q)\widehat{\beta}\,[y]\,\widehat{z}\,R & \text{with } x, \beta \in N(Q) \\
P\widehat{\alpha} \dagger \widehat{x}(Q\widehat{\beta}\,[y]\,\widehat{z}\,R) & \equiv & Q\widehat{\beta}\,[y]\,\widehat{z}(P\widehat{\alpha} \dagger \widehat{x}R) & \text{with } x, z \in N(R)
\end{array}
$$

The rest of the congruence rules are omitted here.

Congruence relation, $\equiv$

- the relation $\equiv$ induces congruence classes on terms
- $cl(P)$ denotes the congruence class of a term P with respect to a relation $\equiv$
- each congruence class has finitely many terms

Restructuring terms

Take for example an arbitrary ${}^*\mathcal{X}$-term of the form :

$$P\widehat{\alpha} \dagger \widehat{x}Q$$

Here, the name α might not be directly accessible. Furthermore, both α and x may occur deep inside their terms. We prove that it is possible to transform the above term - using only congruence rules - to the form

$$C\{P^\alpha\widehat{\alpha} \dagger \widehat{x}Q^x\}$$

where C is a context, and α and x are principal names of $P^\alpha \preccurlyeq P$ and $Q^x \preccurlyeq Q$, respectively.

In other words we can always pick at least one representative of a congruence class $cl(P\widehat{\alpha} \dagger \widehat{x}Q)$, and enable to continue the reduction process. RECALL : $\preccurlyeq$ stands for a subterm relation. $cl(P)$ denotes the congruence class of a term P with respect to $\equiv$.

Restructuring - proof sketch

Lemma - left-side
For every term of the form $P\widehat{\alpha} \dagger \widehat{x}Q$, there exists a context C and a term P^{α}, where α is a principal name for P^{α} and $P^{\alpha} \preccurlyeq P$, such that

$$P\widehat{\alpha} \dagger \widehat{x}Q \equiv C\{P^{\alpha}\widehat{\alpha} \dagger \widehat{x}Q\}$$

Proof : By induction on the structure of P and case analysis.

Lema - right-side
For every term of the form $P\widehat{\alpha} \dagger \widehat{x}Q$, there exists a context C and a term Q^{x}, whose principal name is x and $Q^{x} \preccurlyeq Q$, such that

$$P\widehat{\alpha} \dagger \widehat{x}Q \equiv C\{P\widehat{\alpha} \dagger \widehat{x}Q^{x}\}$$

Proof : By induction on the structure of Q and case analysis.

Restructuring - proof sketch

Theorem
For every term of the form $P\widehat{\alpha} \dagger \widehat{x}Q$ there exists a context C and terms P^α and Q^x whose principal names are α, x respectively. and $P^\alpha \preccurlyeq P$, $Q^x \preccurlyeq Q$, such that

$$P\widehat{\alpha} \dagger \widehat{x}Q \equiv C\{P^\alpha\widehat{\alpha} \dagger \widehat{x}Q^x\}$$

Proof : By using the two previous lemmas, we can construct the proof in two symmetric ways. One is

$$\begin{array}{lcl} P\widehat{\alpha} \dagger \widehat{x}Q & \stackrel{lemma\ 1}{\equiv} & C_1\{P^\alpha\widehat{\alpha} \dagger \widehat{x}Q\} \\ & \stackrel{lemma\ 2}{\equiv} & C_1\{C_2\{P^\alpha\widehat{\alpha} \dagger \widehat{x}Q^x\}\} \\ & \triangleq & C'\{P^\alpha\widehat{\alpha} \dagger \widehat{x}Q^x\}, \quad \text{with} \quad C'\{\ \} = C_1\{C_2\{\ \}\} \end{array}$$

Reduction rules

duality between activation / deactivation

Activation

$$(act\text{-}L) \quad : \quad P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow P\widehat{\alpha} \nearrow\!\!\!\dagger \widehat{x}Q, \quad \alpha \text{ is S-prinipal for } P$$

$$(act\text{-}R) \quad : \quad P\widehat{\alpha} \dagger \widehat{x}Q \rightarrow P\widehat{\alpha} \nwarrow\!\!\!\dagger \widehat{x}Q, \quad x \text{ is S-prinipal for } Q$$

Deactivation

$$(\nearrow\!\!\!\dagger\text{-}deact) \quad : \quad P\widehat{\alpha} \nearrow\!\!\!\dagger \widehat{x}Q \rightarrow P\widehat{\alpha} \dagger \widehat{x}Q, \quad \text{if } \alpha \text{ is L-principal for } P$$

$$(\nwarrow\!\!\!\dagger\text{-}deact) \quad : \quad P\widehat{\alpha} \nwarrow\!\!\!\dagger \widehat{x}Q \rightarrow P\widehat{\alpha} \dagger \widehat{x}Q, \quad \text{if } x \text{ is L-principal for } Q$$

Structural rules

<u>*Left*</u> :

$$
\begin{array}{lcllcl}
(\not\nearrow\text{-}eras) & : & (P \odot \alpha)\widehat{\alpha} \not\nearrow \widehat{x}Q & \to & \mathcal{I}^{Q} \odot \; P \; \odot \mathcal{O}^{Q} \\
(\not\nearrow\text{-}dupl) & : & ([P\rangle\frac{\widehat{\alpha_1}}{\widehat{\alpha_2}}{>}\alpha)\widehat{\alpha} \not\nearrow \widehat{x}Q & \to & P\langle\langle\widehat{\alpha_1}\widehat{\alpha_2} \not\nearrow \widehat{x}Q\rangle\rangle
\end{array}
$$

<u>*Right*</u> :

$$
\begin{array}{lcllcl}
(\not\nwarrow\text{-}eras) & : & P\widehat{\alpha} \not\nwarrow \widehat{x}(x \odot Q) & \to & \mathcal{I}^{P} \odot \; Q \; \odot \mathcal{O}^{P} \\
(\not\nwarrow\text{-}dupl) & : & P\widehat{\alpha} \not\nwarrow \widehat{x}(x{<}\frac{\widehat{x_1}}{\widehat{x_2}}\langle Q]) & \to & \langle\langle P\widehat{\alpha} \not\nwarrow \widehat{x_1}\widehat{x_2}\rangle\rangle Q
\end{array}
$$

Reduction rules

symmetry between left / right rules

Logical rules

$$
\begin{array}{llll}
(\textit{ren-L}) & : \langle y.\alpha\rangle\widehat{\alpha} \dagger \widehat{x}Q & \rightarrow & Q\{y/x\} \\
(\textit{ren-R}) & : P\widehat{\alpha} \dagger \widehat{x}\langle x.\beta\rangle & \rightarrow & P\{\beta/\alpha\} \\
(\textit{ei-insert}) & : (\widehat{y}\,P\,\widehat{\beta}\cdot\alpha)\widehat{\alpha} \dagger \widehat{x}(Q\,\widehat{\gamma}\,[x]\,\widehat{z}\,R) & \rightarrow & \begin{cases} (Q\widehat{\gamma} \dagger \widehat{y}P)\widehat{\beta} \dagger \widehat{z}R \\ Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\beta} \dagger \widehat{z}R) \end{cases}
\end{array}
$$

Operational properties

Theorem - preservation of free names, by $\equiv$ and $\rightarrow$

If $P \equiv Q$ then $\mathrm{N}(P) = \mathrm{N}(Q)$
If $P \rightarrow Q$ then $\mathrm{N}(P) = \mathrm{N}(Q)$

Theorem - preservation of linearity, by $\equiv$ and $\rightarrow$

If P is linear and $P \equiv Q$ then Q is linear
If P is linear and $P \rightarrow Q$ then Q is linear

Proof : By treating carefully each rule.

Operational properties

Theorem - preservation of types, by $\equiv$ and $\rightarrow$

Let S be an ${}^{\copyright}\mathcal{X}$-term and Γ, Δ contexts. Then the following holds :

$$\begin{array}{l} \text{If } S :\cdot \Gamma \vdash \Delta \text{ and } S \equiv S', \text{ then } S' :\cdot \Gamma \vdash \Delta \\ \text{If } S :\cdot \Gamma \vdash \Delta \text{ and } S \rightarrow S', \text{ then } S' :\cdot \Gamma \vdash \Delta \end{array}$$

Proof : By checking that this holds for all equations and reduction rules.

Theorem - strong normalisation [3]

The reduction system of ${}^{*}\mathcal{X}$ is strongly normalising on simply-typed terms.

Proof : Preservation of strong normalisation, based on SN of $\mathcal{X}$ and ${}^{*}\mathcal{X}$.

3. Not yet formally proved

Sources

- You may check a draft article at this URL :
  https://sites.google.com/site/dragisazunic/Home/research/CX.pdf
- Also, my PhD thesis available at :
  http://tel.archives-ouvertes.fr/tel-00265549

Thank you