Presheaf models for guarded recursion

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March 4, 2013

Overview

- Guarded recursion applications
  - Computing with streams
  - Modelling higher order store
- Model: the topos of trees
- Recursive types as fixed points on universes
- Intensional models

Motivation

Motivation 1: Computing with streams

- Which of the following streams are well-defined:

```
zeros = 0:zeros
xs = xs
```

- Much less obvious when higher types are involved

$$\mathtt{mergef}:(\mathtt{int} \to \mathtt{int} \to \mathtt{S(int)} \to \mathtt{S(int)}) \to \mathtt{S(int)} \to \mathtt{S(int)} \to \mathtt{S(int)}$$

```
mergef f x:xs y:ys = f x y (mergef f xs ys)
```

- A nonproductive example:

```
badf x y xs = xs
mergef badf x:xs y:ys = mergef xs ys
```

- (example due to Bob Atkey)

Capturing productivity in types

- Introduce modal operator $\blacktriangleright$

$$\begin{aligned}
\mathtt{S(int)} \;\mathtt{=}\; & \mu\mathtt{X.int} \times \blacktriangleright X \\
\mathtt{hd} : & \; \mathtt{S(int)} \to \mathtt{int} \\
\mathtt{tail} : & \; \mathtt{S(int)} \to \blacktriangleright \mathtt{S(int)} \\
\mathtt{cons} : & \; \mathtt{int} \times \blacktriangleright \mathtt{S(int)} \to \mathtt{S(int)}
\end{aligned}$$

- Fixed points

$$\mathtt{fix} : \big(\blacktriangleright \mathtt{S(int)} \to \mathtt{S(int)}\big) \to \mathtt{S(int)}$$

$$\mathtt{zeros} \;\mathtt{=}\; \mathtt{fix}(\lambda\mathtt{xs.0{:}xs})$$

Capturing productivity in types

- $\blacktriangleright$ is applicative functor

$$\mathtt{next} : \mathtt{X} \to \blacktriangleright \mathtt{X}$$
$$\circledast : \blacktriangleright (\mathtt{X} \to \mathtt{Y}) \to \blacktriangleright \mathtt{X} \to \blacktriangleright \mathtt{Y}$$

- Typing `mergef`

$$\begin{aligned}\mathtt{mergef} :& (\mathtt{int} \to \mathtt{int} \to \blacktriangleright \mathtt{S(int)} \to \mathtt{S(int)}) \\ & \to \mathtt{S(int)} \to \mathtt{S(int)} \to \mathtt{S(int)}\end{aligned}$$
$$\mathtt{mergef\ f\ =\ fix}(\lambda \mathtt{g}.\lambda(\mathtt{x{:}xs})\lambda(\mathtt{y{:}ys}).\mathtt{f\ x\ y\ (g} \circledast \mathtt{xs} \circledast \mathtt{ys}))$$

- where

$$\mathtt{g} : \blacktriangleright(\mathtt{S(int)} \to \mathtt{S(int)} \to \mathtt{S(int)})$$

- Note: `mergef badf` is not well typed

Motivation 2: Modelling higher-order store

- Idea: interpret types as subsets of Value
- except the subset should depend on the world
- Would like to solve (but can not)

$$\mathcal{W} = N \to_{\mathrm{fin}} \mathcal{T} \qquad \mathcal{T} = \mathcal{W} \to_{\mathrm{mon}} \mathcal{P}(\mathrm{Value})$$

- Can solve this equation

$$\widehat{\mathcal{T}} = \mu X.\ \blacktriangleright((N \to_{\mathrm{fin}} X) \to_{\mathrm{mon}} \mathcal{P}(\mathrm{Value}))$$

- Note that these involve non-standard function spaces
- Can model higher-order store in expressive type theory with guarded recursion

The topos of trees

The topos of trees

- $\mathcal{S} = \mathbf{Set}^{\omega^{\mathrm{op}}}$
- Objects

$$X(1) \xleftarrow{r_1} X(2) \xleftarrow{r_2} X(3) \longleftarrow \dots$$

- Morphism

$$\begin{tikzcd}
X(1) \arrow[d, "f_1"'] & X(2) \arrow[l] \arrow[d, "f_2"'] & X(3) \arrow[l] \arrow[d, "f_3"'] & \dots \arrow[l] \\
Y(1) & Y(2) \arrow[l] & Y(3) \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

- Example: object of streams of integers `S(int)`

$$\mathbb{Z} \xleftarrow{\ \pi\ } \mathbb{Z}^2 \xleftarrow{\ \pi\ } \mathbb{Z}^3 \xleftarrow{\ \pi\ } \dots$$

- For $x \in X(m)$ define

$$x|_n = r_n \circ \cdots \circ r_{m-1}(x).$$

An endofunctor

- Define $\blacktriangleright X$ ("later X")

$$\{*\} \longleftarrow X(1) \longleftarrow X(2) \longleftarrow \ldots$$

- Preserves limits but not colimits
- Note that $\mathtt{S(int)} \cong \mathbb{Z} \times \blacktriangleright \mathtt{S(int)}$:

$$\mathbb{Z} \times 1 \xleftarrow{\mathbb{Z}\times !} \mathbb{Z} \times \mathbb{Z} \xleftarrow{\mathbb{Z} \times \pi} \mathbb{Z} \times \mathbb{Z}^2 \xleftarrow{\mathbb{Z} \times \pi} \ldots$$

- Define $\mathrm{next} : X \to \blacktriangleright X$

$$\begin{tikzcd}
X(1) \arrow[d] & X(2) \arrow[l, "r_1"'] \arrow[d, "r_1"'] & X(3) \arrow[l, "r_2"'] \arrow[d, "r_2"'] & \ldots \arrow[l] \\
\{*\} & X(1) \arrow[l] & X(2) \arrow[l, "r_1"'] & \ldots \arrow[l]
\end{tikzcd}$$

Fixed points

- A morphism factoring through next is called *contractive*

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{next}"] \arrow[dr, "f"'] & \blacktriangleright X \arrow[d, "\exists"] \\
& X
\end{tikzcd}
$$

- Contractive morphisms have *unique* fixed points
- Fixed point operator

$$\mathrm{fix}_X : (\blacktriangleright X \to X) \to X$$

Construction of fixed points

- Given $f : \blacktriangleright X \to X$:

$$
\begin{tikzcd}
\{*\} \arrow[d, "f_1"'] & X(1) \arrow[l] \arrow[d, "f_2"'] & X(2) \arrow[l, "r_1"'] \arrow[d, "f_3"'] & \dots \arrow[l] \\
X(1) & X(2) \arrow[l, "r_1"'] & X(3) \arrow[l, "r_2"'] & \dots \arrow[l]
\end{tikzcd}
$$

- Construct $\mathrm{fix}_X(f) : 1 \to X$:

$$
\begin{tikzcd}
1 \arrow[d, "f_1"] & 1 \arrow[l] \arrow[d, "f_2 \circ f_1"] & 1 \arrow[l] \arrow[d, "f_3 \circ f_2 \circ f_1"] & \dots \arrow[l] \\
X(1) & X(2) \arrow[l, "r_1"'] & X(3) \arrow[l, "r_2"'] & \dots \arrow[l]
\end{tikzcd}
$$

Recursive types

Example

- Consider type constructor F

$$FX = \blacktriangleright X \times \mathbb{Z}$$

- $F(\mathtt{S(int)}) \cong \mathtt{S(int)}$

$$\begin{array}{llllllll}
X & X(1) \longleftarrow & X(2) \longleftarrow & X(3) \longleftarrow & X(4) \ldots \\
FX & 1 \times \mathbb{Z} \longleftarrow & X(1) \times \mathbb{Z} \longleftarrow & X(2) \times \mathbb{Z} \longleftarrow & X(3) \times \mathbb{Z} \ldots \\
F^2X & 1 \times \mathbb{Z} \longleftarrow & 1 \times \mathbb{Z}^2 \longleftarrow & X(1) \times \mathbb{Z}^2 \longleftarrow & X(2) \times \mathbb{Z}^2 \ldots
\end{array}$$

- If $n \geq k$ then $F^n(X)(k) = 1 \times \mathbb{Z}^k \cong \mathtt{S(int)}(k)$

Recursive domain equations

- Recall $F : \mathcal{S} \to \mathcal{S}$ *strong* if exists

$$F_{X,Y} : Y^X \to FY^{FX}$$

- Say F locally contractive if each $F_{X,Y}$ contractive:

$$Y^X \xrightarrow{\text{next}} \blacktriangleright(Y^X) \xrightarrow{G_{X,Y}} FY^{FX}$$

  and $G_{X,Y}$ respects composition and identity
- Generalises to mixed variance functors of many variables
- **Theorem:** If $F : \mathcal{S}^{\mathrm{op}} \times \mathcal{S} \to \mathcal{S}$ is locally contractive then there exists X such that $F(X,X) \cong X$. Moreover, X unique up to isomorphism
- Solutions are initial dialgebras

Proof of algebraic compactness (sketch)

- First: existence of solutions to covariant equations
- **Lemma:** Suppose $F : \mathcal{S} \to \mathcal{S}$ is locally contractive and $n \geq k$. Then $F^n(X)(k) \cong F^n(Y)(k)$ for all X, Y.
- Construct solution to F as diagonal of

$$
\begin{tikzcd}
F(1)(1) & F^2(1)(1) \arrow[l, "F!_1"'] & F^3(1)(1) \arrow[l, "F^2(!)_1"'] & F^4(1)(1) \arrow[l, "F^3(!)_1"'] & \dots \arrow[l] \\
F(1)(2) \arrow[u] & F^2(1)(2) \arrow[u] \arrow[l, "F!_2"'] & F^3(1)(2) \arrow[u] \arrow[l, "F^2(!)_2"'] & F^4(1)(2) \arrow[u] \arrow[l, "F^3(!)_2"'] & \dots \arrow[l] \\
F(1)(3) \arrow[u] & F^2(1)(3) \arrow[u] \arrow[l, "F!_3"'] & F^3(1)(3) \arrow[u] \arrow[l, "F^2(!)_3"'] & F^4(1)(3) \arrow[u] \arrow[l, "F^3(!)_3"'] & \dots \arrow[l] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

Proof of algebraic compactness

- All solutions to $F(X) \cong X$ are initial algebras and final coalgebras
- because $h : X \to Y$ is algebra map from $F(X) \cong X$ to g iff

$$\begin{tikzcd} F(X) \arrow[d, "F(h)"'] & X \arrow[l, "\cong"'] \arrow[d, "h"] \\ F(Y) \arrow[r, "g"] & Y \end{tikzcd}$$

  i.e. iff h fixed point of contractive map
- Solutions to general recursive domain equations from Freyd's theory of algebraic compactness

Universes

Universes in type theory

- Universe type U : Type

$$\frac{\Gamma \vdash A : U}{\Gamma \vdash \mathrm{El}(A) : \mathsf{Type}}$$

- Basic elements $\mathbb{N} : U$, $\mathrm{El}(\mathbb{N}) = \mathbb{N}$

$$\frac{\Gamma \vdash A : U \quad \Gamma \vdash B : U}{\Gamma \vdash A \times B : U}$$

$$\mathrm{El}(A \times B) = \mathrm{El}(A) \times \mathrm{El}(B)$$

Guarded recursion

- Universe closed under $\blacktriangleright$

$$\frac{\Gamma \vdash A : \blacktriangleright U}{\Gamma \vdash \triangleright A : U}$$

$$\mathrm{El}(\triangleright(\mathrm{next}(A))) = \blacktriangleright \mathrm{El}(A)$$

- Type of streams as fixed point

$$\mathtt{S(int)} = \mathrm{fix}(\lambda X : \blacktriangleright U . \mathbb{N} \times \triangleright X)$$

- Then

$$\begin{aligned} \mathrm{El}(\mathtt{S(int)}) &= \mathrm{El}(\mathbb{N} \times \triangleright(\mathrm{next}(\mathtt{S(int)}))) \\ &= \mathbb{N} \times \blacktriangleright \mathrm{El}(\mathtt{S(int)}) \end{aligned}$$

A universe in $\mathbf{Set}^{\omega^{\mathrm{op}}}$

- Assume given universe U in **Set**
- Construct universe V in $\mathbf{Set}^{\omega^{\mathrm{op}}}$

$$V(1) \xleftarrow{r_1^V} V(2) \xleftarrow{r_2^V} V(3) \xleftarrow{r_3^V} V(4) \longleftarrow \ldots$$

- Define $V(1) = U$

$$V(n+1) = \{(X_1, \ldots, X_{n+1}, f_1 \ldots f_n) \mid \forall i. X_i \in U, f_i : X_{i+1} \to X_i\}$$
$$r_n^V(X_1, \ldots, X_{n+1}, f_1 \ldots f_n) = (X_1, \ldots, X_n, f_1 \ldots f_{n-1})$$

- (Construction due to Hofmann and Streicher)

Global elements of the universe

- Maps $1 \to V$

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[l] \arrow[d] & 1 \arrow[l] \arrow[d] & 1 \arrow[l] \arrow[d] & \dots \arrow[l] \\
V(1) & V(2) \arrow[l] & V(3) \arrow[l] & V(4) \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

- Correspond to objects

$$X(1) \longleftarrow X(2) \longleftarrow X(3) \longleftarrow X(4) \longleftarrow \dots$$

- Such that $X(n) \in U$
- (Generalises to statement about dependent types)

Later operator

- Need $\triangleright : \blacktriangleright V \to V$

$$
\begin{tikzcd}
1 \arrow[d] & V(1) \arrow[l] \arrow[d] & V(2) \arrow[l] \arrow[d] & V(3) \arrow[l] \arrow[d] & \dots \arrow[l] \\
V(1) & V(2) \arrow[l] & V(3) \arrow[l] & V(4) \arrow[l] & \dots \arrow[l]
\end{tikzcd}
$$

- Define

$$\underset{1}{\triangleright}(\star) = 1$$

$$\underset{n+1}{\triangleright}(X_1 \dots X_n, f_1 \dots f_{n-1}) = (1, X_1 \dots X_n, !, f_1 \dots f_{n-1})$$

- If A corresponds to $\bar{A} : 1 \to V$ then

$$1 \xrightarrow{\bar{A}} V \xrightarrow{\mathrm{next}} \blacktriangleright V \xrightarrow{\triangleright} V$$

corresponds to $\blacktriangleright A$.

Recursive type example

- Consider type constructor F

$$FX = \blacktriangleright X \times \mathbb{Z}$$

- $\bar{F} : \blacktriangleright V \to V$

$$\begin{aligned} \bar{F}_1(\star) &= 1 \times \mathbb{Z} \\ \bar{F}_{n+1}(X_1 \dots X_n, f_1 \dots f_{n-1}) &= (1 \times \mathbb{Z}, X_1 \times \mathbb{Z} \dots X_n \times \mathbb{Z}, \\ &\qquad ! \times id, f_1 \times id \dots f_{n-1} \times id) \end{aligned}$$

- $\mathrm{fix}(\bar{F}) : 1 \to V$, n'th component is

$$1 \xrightarrow{\bar{F}_1} V(1) \xrightarrow{\bar{F}_2} V(2) \quad \dots \xrightarrow{\bar{F}_n} V(n)$$

Intensional models

Type constructors vs universe maps

- Are type constructors the same as universe maps $U \to U$?
- Type constructors F in mathematics are usually given up to isomorphism and preserve isomorphism classes
- These induce maps $\bar{F} : U \to U$

$$F(\mathrm{fix}(\bar{F})) \cong \bar{F}(\mathrm{fix}(\bar{F})) = \mathrm{fix}(\bar{F})$$

- How do we prove uniqueness of solutions for F?
- A map $U \to U$ may not preserve isomorphism classes
- **Theorem.** In groupoid model, let U be universe of small discrete groupoids, then maps $U \to U$ correspond to type constructors

Intensional models

- **Theorem** (Shulman): If $\mathbb{C}$ is a model of intensional type theory, so is $\mathbb{C}^{\omega^{\mathrm{op}}}$.
- **Theorem**: $\mathbb{C}^{\omega^{\mathrm{op}}}$ models guarded recursion plus the rule

$$\frac{\Gamma, f : \blacktriangleright A \to A, y : A \vdash p : \mathrm{Id}_A(y, f(\mathrm{next}(y)))}{\Gamma, f : \blacktriangleright A \to A, y : A \vdash \mathrm{UFP}(p) : \mathrm{Id}_A(y, \mathrm{fix}\, x.f(x))}$$

The groupoid model

- Let **Grpd** be the category of groupoids, and let V be the universe of discrete groupoids in $\mathbf{Grpd}^{\omega^{\mathrm{op}}}$
- **Theorem**: Let $\bar{F} : V \to V$ a code for a type constructor F. Let A be a fibrant object and $\bar{A} : 1 \to V$ be a code of A. Then the following are equivalent
  - A is a fixed point of F up to isomorphism
  - $\bar{A}$ is a fixed point of $\bar{F}$ up to internal equality
- **Corollary**: Let $F : \mathbf{Grpd}^{\omega^{\mathrm{op}}} \to \mathbf{Grpd}^{\omega^{\mathrm{op}}}$ be a type constructor. If F has a contractive code $\bar{F} : V \to V$ then it has a unique fixed point up to isomorphism

Conclusions

- Guarded recursion and $\blacktriangleright$ gives new approach to coinductive types in type theory
- Guarded recursive types useful in modelling higher-order store
- Guarded recursion can be modelled in a presheaf model
- Universes allow recursive type definitions to be reduced to fixed points on term level
- In groupoid model maps $U \to U$ correspond to type constructors

Future work

- Extending Agda with guarded recursion
- Guarded recursive types vs coinductive types
- Denotational semantics in synthetic guarded domain theory
- Other models