

# Presheaf models for guarded recursion

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# Overview

- Guarded recursion applications
  - Computing with streams
  - Modelling higher order store
- Model: the topos of trees
- Recursive types as fixed points on universes
- Intensional models

Motivation

## Motivation 1: Computing with streams

- Which of the following streams are well-defined:

$$\text{zeros} = 0:\text{zeros}$$
$$\text{xs} = \text{xs}$$

- Much less obvious when higher types are involved

$$\text{mergef}:(\text{int} \rightarrow \text{int} \rightarrow \text{S}(\text{int}) \rightarrow \text{S}(\text{int}))$$
$$\rightarrow \text{S}(\text{int}) \rightarrow \text{S}(\text{int}) \rightarrow \text{S}(\text{int})$$
$$\text{mergef } f \text{ } x:\text{xs} \text{ } y:\text{ys} = f \text{ } x \text{ } y \text{ } (\text{mergef } f \text{ } \text{xs} \text{ } \text{ys})$$

- A nonproductive example:

$$\text{badf } x \text{ } y \text{ } \text{xs} = \text{xs}$$
$$\text{mergef } \text{badf } \text{ } x:\text{xs} \text{ } y:\text{ys} = \text{mergef } \text{xs} \text{ } \text{ys}$$

- (example due to Bob Atkey)

## Capturing productivity in types

- Introduce modal operator  $\blacktriangleright$

$$S(\text{int}) = \mu X. \text{int} \times \blacktriangleright X$$

$$\text{hd}: S(\text{int}) \rightarrow \text{int}$$

$$\text{tail}: S(\text{int}) \rightarrow \blacktriangleright S(\text{int})$$

$$\text{cons}: \text{int} \times \blacktriangleright S(\text{int}) \rightarrow S(\text{int})$$

- Fixed points

$$\text{fix}: (\blacktriangleright S(\text{int}) \rightarrow S(\text{int})) \rightarrow S(\text{int})$$

$$\text{zeros} = \text{fix}(\lambda xs. 0:xs)$$

## Capturing productivity in types

- $\blacktriangleright$  is applicative functor

$$\text{next} : X \rightarrow \blacktriangleright X$$
$$\otimes : \blacktriangleright (X \rightarrow Y) \rightarrow \blacktriangleright X \rightarrow \blacktriangleright Y$$

- Typing mergef

$$\text{mergef} : (\text{int} \rightarrow \text{int} \rightarrow \blacktriangleright S(\text{int}) \rightarrow S(\text{int}))$$
$$\rightarrow S(\text{int}) \rightarrow S(\text{int}) \rightarrow S(\text{int})$$
$$\text{mergef } f = \text{fix}(\lambda g. \lambda (x:xs) \lambda (y:ys). f \ x \ y \ (g \otimes \ xs \ \otimes \ ys))$$

- where

$$g : \blacktriangleright (S(\text{int}) \rightarrow S(\text{int}) \rightarrow S(\text{int}))$$

- Note: mergef badf is not well typed

## Motivation 2: Modelling higher-order store

- Idea: interpret types as subsets of Value
- except the subset should depend on the world
- Would like to solve (but can not)

$$\mathcal{W} = N \rightarrow_{\text{fin}} \mathcal{T} \quad \mathcal{T} = \mathcal{W} \rightarrow_{\text{mon}} \mathcal{P}(\text{Value})$$

- Can solve this equation

$$\widehat{\mathcal{T}} = \mu X. \blacktriangleright((N \rightarrow_{\text{fin}} X) \rightarrow_{\text{mon}} \mathcal{P}(\text{Value}))$$

- Note that these involve non-standard function spaces
- Can model higher-order store in expressive type theory with guarded recursion

The topos of trees



# The topos of trees

- $\mathcal{S} = \mathbf{Set}^{\omega^{\text{op}}}$
- Objects

$$X(1) \xleftarrow{r_1} X(2) \xleftarrow{r_2} X(3) \xleftarrow{\quad} \dots$$

- Morphism

$$\begin{array}{ccccccc} X(1) & \longleftarrow & X(2) & \longleftarrow & X(3) & \longleftarrow & \dots \\ f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & \\ Y(1) & \longleftarrow & Y(2) & \longleftarrow & Y(3) & \longleftarrow & \dots \end{array}$$

- Example: object of streams of integers  $\mathcal{S}(\text{int})$

$$\mathbb{Z} \xleftarrow{\pi} \mathbb{Z}^2 \xleftarrow{\pi} \mathbb{Z}^3 \xleftarrow{\pi} \dots$$

- For  $x \in X(m)$  define

$$x|_n = r_n \circ \dots \circ r_{m-1}(x).$$

## An endofunctor

- Define  $\blacktriangleright X$  (“later  $X$ ”)

$$\{*\} \longleftarrow X(1) \longleftarrow X(2) \longleftarrow \dots$$

- Preserves limits but not colimits
- Note that  $S(\text{int}) \cong \mathbb{Z} \times \blacktriangleright S(\text{int})$ :

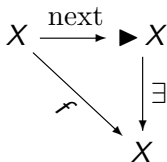
$$\mathbb{Z} \times 1 \xleftarrow{\mathbb{Z} \times !} \mathbb{Z} \times \mathbb{Z} \xleftarrow{\mathbb{Z} \times \pi} \mathbb{Z} \times \mathbb{Z}^2 \xleftarrow{\mathbb{Z} \times \pi} \dots$$

- Define  $\text{next} : X \rightarrow \blacktriangleright X$

$$\begin{array}{ccccccc} X(1) & \xleftarrow{r_1} & X(2) & \xleftarrow{r_2} & X(3) & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ \{*\} & \longleftarrow & X(1) & \xleftarrow{r_1} & X(2) & \longleftarrow & \dots \end{array}$$

# Fixed points

- A morphism factoring through `next` is called *contractive*



- Contractive morphisms have *unique* fixed points
- Fixed point operator

$$\text{fix}_X : (\blacktriangleright X \rightarrow X) \rightarrow X$$

## Construction of fixed points

- Given  $f : \mathbf{1} \rightarrow X$ :

$$\begin{array}{ccccccc} \{*\} & \longleftarrow & X(1) & \xleftarrow{r_1} & X(2) & \longleftarrow & \dots \\ f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & \\ X(1) & \xleftarrow{r_1} & X(2) & \xleftarrow{r_2} & X(3) & \longleftarrow & \dots \end{array}$$

- Construct  $\text{fix}_X(f) : \mathbf{1} \rightarrow X$ :

$$\begin{array}{ccccccc} \mathbf{1} & \longleftarrow & \mathbf{1} & \longleftarrow & \mathbf{1} & \longleftarrow & \dots \\ \downarrow f_1 & & \downarrow f_2 \circ f_1 & & \downarrow f_3 \circ f_2 \circ f_1 & & \\ X(1) & \xleftarrow{r_1} & X(2) & \xleftarrow{r_2} & X(3) & \longleftarrow & \dots \end{array}$$

Recursive types

## Example

- Consider type constructor  $F$

$$FX = \blacktriangleright X \times \mathbb{Z}$$

- $F(S(\text{int})) \cong S(\text{int})$

$$X \quad X(1) \longleftarrow X(2) \longleftarrow X(3) \longleftarrow X(4) \dots$$

$$FX \quad 1 \times \mathbb{Z} \longleftarrow X(1) \times \mathbb{Z} \longleftarrow X(2) \times \mathbb{Z} \longleftarrow X(3) \times \mathbb{Z} \dots$$

$$F^2X \quad 1 \times \mathbb{Z} \longleftarrow 1 \times \mathbb{Z}^2 \longleftarrow X(1) \times \mathbb{Z}^2 \longleftarrow X(2) \times \mathbb{Z}^2 \dots$$

- If  $n \geq k$  then  $F^n(X)(k) = 1 \times \mathbb{Z}^k \cong S(\text{int})(k)$

## Recursive domain equations

- Recall  $F : \mathcal{S} \rightarrow \mathcal{S}$  *strong* if exists

$$F_{X,Y} : Y^X \rightarrow FY^{FX}$$

- Say  $F$  locally contractive if each  $F_{X,Y}$  contractive:

$$Y^X \xrightarrow{\text{next}} \blacktriangleright (Y^X) \xrightarrow{G_{X,Y}} FY^{FX}$$

and  $G_{X,Y}$  respects composition and identity

- Generalises to mixed variance functors of many variables
- Theorem:** If  $F : \mathcal{S}^{\text{op}} \times \mathcal{S} \rightarrow \mathcal{S}$  is locally contractive then there exists  $X$  such that  $F(X, X) \cong X$ . Moreover,  $X$  unique up to isomorphism
- Solutions are initial dialgebras

## Proof of algebraic compactness (sketch)

- First: existence of solutions to covariant equations
- **Lemma:** Suppose  $F : \mathcal{S} \rightarrow \mathcal{S}$  is locally contractive and  $n \geq k$ . Then  $F^n(X)(k) \cong F^n(Y)(k)$  for all  $X, Y$ .
- Construct solution to  $F$  as diagonal of

$$\begin{array}{ccccccc} F(1)(1) & \xleftarrow{F^!_1} & F^2(1)(1) & \xleftarrow{F^{2(!)}_1} & F^3(1)(1) & \xleftarrow{F^{3(!)}_1} & F^4(1)(1) \xleftarrow{\dots} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ F(1)(2) & \xleftarrow{F^!_2} & F^2(1)(2) & \xleftarrow{F^{2(!)}_2} & F^3(1)(2) & \xleftarrow{F^{3(!)}_2} & F^4(1)(2) \xleftarrow{\dots} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ F(1)(3) & \xleftarrow{F^!_3} & F^2(1)(3) & \xleftarrow{F^{2(!)}_3} & F^3(1)(3) & \xleftarrow{F^{3(!)}_3} & F^4(1)(3) \xleftarrow{\dots} \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$



## Proof of algebraic compactness

- All solutions to  $F(X) \cong X$  are initial algebras and final coalgebras
- because  $h : X \rightarrow Y$  is algebra map from  $F(X) \cong X$  to  $g$  iff

$$\begin{array}{ccc} F(X) & \xleftarrow{\cong} & X \\ F(h) \downarrow & & \downarrow h \\ F(Y) & \xrightarrow{g} & Y \end{array}$$

i.e. iff  $h$  fixed point of contractive map

- Solutions to general recursive domain equations from Freyd's theory of algebraic compactness

Universes

# Universes in type theory

- Universe type  $U : \text{Type}$

$$\frac{\Gamma \vdash A : U}{\Gamma \vdash \text{El}(A) : \text{Type}}$$

- Basic elements  $\mathbb{N} : U, \text{El}(\mathbb{N}) = \mathbb{N}$

$$\frac{\Gamma \vdash A : U \quad \Gamma \vdash B : U}{\Gamma \vdash A \times B : U}$$

$$\text{El}(A \times B) = \text{El}(A) \times \text{El}(B)$$

## Guarded recursion

- Universe closed under  $\blacktriangleright$

$$\frac{\Gamma \vdash A : \blacktriangleright U}{\Gamma \vdash \triangleright A : U}$$

$$\text{El}(\triangleright(\text{next}(A))) = \blacktriangleright \text{El}(A)$$

- Type of streams as fixed point

$$\text{S}(\text{int}) = \text{fix}(\lambda X : \blacktriangleright U. \mathbb{N} \times \triangleright X)$$

- Then

$$\begin{aligned} \text{El}(\text{S}(\text{int})) &= \text{El}(\mathbb{N} \times \triangleright(\text{next}(\text{S}(\text{int})))) \\ &= \mathbb{N} \times \blacktriangleright \text{El}(\text{S}(\text{int})) \end{aligned}$$

## A universe in $\mathbf{Set}^{\omega^{\text{op}}}$

- Assume given universe  $U$  in  $\mathbf{Set}$
- Construct universe  $V$  in  $\mathbf{Set}^{\omega^{\text{op}}}$

$$V(1) \xleftarrow{r_1^V} V(2) \xleftarrow{r_2^V} V(3) \xleftarrow{r_3^V} V(4) \longleftarrow \dots$$

- Define  $V(1) = U$

$$V(n+1) = \{(X_1, \dots, X_{n+1}, f_1 \dots f_n) \mid \forall i. X_i \in U, f_i : X_{i+1} \rightarrow X_i\}$$

$$r_n^V(X_1, \dots, X_{n+1}, f_1 \dots f_n) = (X_1, \dots, X_n, f_1 \dots f_{n-1})$$

- (Construction due to Hofmann and Streicher)

## Global elements of the universe

- Maps  $1 \rightarrow V$

$$\begin{array}{ccccccc} 1 & \longleftarrow & 1 & \longleftarrow & 1 & \longleftarrow & 1 & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ V(1) & \longleftarrow & V(2) & \longleftarrow & V(3) & \longleftarrow & V(4) & \longleftarrow & \dots \end{array}$$

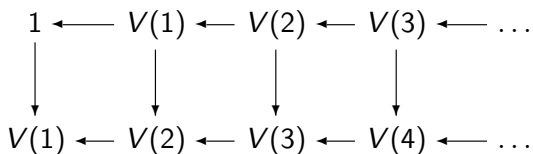
- Correspond to objects

$$X(1) \longleftarrow X(2) \longleftarrow X(3) \longleftarrow X(4) \longleftarrow \dots$$

- Such that  $X(n) \in U$
- (Generalises to statement about dependent types)

## Later operator

- Need  $\triangleright : \blacktriangleright V \rightarrow V$



- Define

$$\begin{aligned} \triangleright_1(\star) &= 1 \\ \triangleright_{n+1}(X_1 \dots X_n, f_1 \dots f_{n-1}) &= (1, X_1 \dots X_n, !, f_1 \dots f_{n-1}) \end{aligned}$$

- If  $A$  corresponds to  $\bar{A} : 1 \rightarrow V$  then

$$1 \xrightarrow{\bar{A}} V \xrightarrow{\text{next}} \blacktriangleright V \xrightarrow{\triangleright} V$$

corresponds to  $\blacktriangleright A$ .

## Recursive type example

- Consider type constructor  $F$

$$FX = \blacktriangleright X \times \mathbb{Z}$$

- $\bar{F} : \blacktriangleright V \rightarrow V$

$$\bar{F}_1(\star) = 1 \times \mathbb{Z}$$

$$\bar{F}_{n+1}(X_1 \dots X_n, f_1 \dots f_{n-1}) = (1 \times \mathbb{Z}, X_1 \times \mathbb{Z} \dots X_n \times \mathbb{Z}, \\ ! \times id, f_1 \times id \dots f_{n-1} \times id)$$

- $\text{fix}(\bar{F}) : 1 \rightarrow V$ ,  $n$ 'th component is

$$1 \xrightarrow{\bar{F}_1} V(1) \xrightarrow{\bar{F}_2} V(2) \quad \dots \xrightarrow{\bar{F}_n} V(n)$$



Intensional models

## Type constructors vs universe maps

- Are type constructors the same as universe maps  $U \rightarrow U$ ?
- Type constructors  $F$  in mathematics are usually given up to isomorphism and preserve isomorphism classes
- These induce maps  $\bar{F} : U \rightarrow U$

$$F(\text{fix}(\bar{F})) \cong \bar{F}(\text{fix}(\bar{F})) = \text{fix}(\bar{F})$$

- How do we prove uniqueness of solutions for  $F$ ?
- A map  $U \rightarrow U$  may not preserve isomorphism classes
- **Theorem.** In groupoid model, let  $U$  be universe of small discrete groupoids, then maps  $U \rightarrow U$  correspond to type constructors

# Intensional models

- **Theorem** (Shulman): If  $\mathbb{C}$  is a model of intensional type theory, so is  $\mathbb{C}^{\omega^{\text{op}}}$ .
- **Theorem:**  $\mathbb{C}^{\omega^{\text{op}}}$  models guarded recursion plus the rule

$$\frac{\Gamma, f : \blacktriangleright A \rightarrow A, y : A \vdash p : \text{Id}_A(y, f(\text{next}(y)))}{\Gamma, f : \blacktriangleright A \rightarrow A, y : A \vdash \text{UFP}(p) : \text{Id}_A(y, \text{fix } x.f(x))}$$

# The groupoid model

- Let **Grpd** be the category of groupoids, and let  $V$  be the universe of discrete groupoids in  $\mathbf{Grpd}^{\omega^{\text{op}}}$
- **Theorem:** Let  $\bar{F} : V \rightarrow V$  a code for a type constructor  $F$ . Let  $A$  be a fibrant object and  $\bar{A} : 1 \rightarrow V$  be a code of  $A$ . Then the following are equivalent
  - $A$  is a fixed point of  $F$  up to isomorphism
  - $\bar{A}$  is a fixed point of  $\bar{F}$  up to internal equality
- **Corollary:** Let  $F : \mathbf{Grpd}^{\omega^{\text{op}}} \rightarrow \mathbf{Grpd}^{\omega^{\text{op}}}$  be a type constructor. If  $F$  has a contractive code  $\bar{F} : V \rightarrow V$  then it has a unique fixed point up to isomorphism

# Conclusions

- Guarded recursion and  $\blacktriangleright$  gives new approach to coinductive types in type theory
- Guarded recursive types useful in modelling higher-order store
- Guarded recursion can be modelled in a presheaf model
- Universes allow recursive type definitions to be reduced to fixed points on term level
- In groupoid model maps  $U \rightarrow U$  correspond to type constructors

## Future work

- Extending Agda with guarded recursion
- Guarded recursive types vs coinductive types
- Denotational semantics in synthetic guarded domain theory
- Other models