

Mechanizing Metatheory with LF and Twelf

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CADE Twelf Tutorial

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(Modified from 2009 POPL tutorial by the CMU POP Group)

What We'll Learn

Representation of languages and logics in LF

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- Syntax

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Representation of languages and logics in LF

- Syntax $e ::= x \mid \lambda x.e \mid e_1 e_2$

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- Syntax
- Judgements

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Typing : $\Gamma \vdash e : \tau$

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Operational semantics : $e \mapsto e'$

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Translations between languages; compiler passes

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Proof theory for a logic

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Mechanization of metatheory using Twelf

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Type safety

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Mechanization of metatheory using Twelf

Progress: If $e : \tau$, then e *value* or else $e \mapsto e'$ (for some e').

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Representation of languages and logics in LF

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Mechanization of metatheory using Twelf

Decidability of type checking

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Cut elimination for a logic

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Correctness of compiler transformations

Part I

Basic Twelf Skills

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- ① Representing syntax and judgements
- ② Totality of judgements
- ③ Proving metatheorems

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What is LF?

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- Now: basic abstract syntax and judgements
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A language is **inductively** presented by a collection of **generators**, whose types are specified by a **signature**.

Natural Numbers

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The judgement $n \mathbf{nat}$ is the **strongest** (most restrictive) judgement **closed under** (obeying) these rules.

Abstract Syntax in LF

$$\frac{}{\text{zero } \mathbf{nat}} \qquad \frac{n \mathbf{nat}}{\text{succ}(n) \mathbf{nat}}$$

Methodology:

- Syntactic category becomes an LF type
- Each rule becomes a generator in the LF signature.

In Twelf:

```
nat  : type.  
zero : nat.  
succ : nat -> nat.
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LF Representation

```
nat  : type.  
zero : nat.  
succ : nat -> nat.
```

Construct terms by applying generators:

```
1 : nat = succ zero  
2 : nat = succ (succ zero)
```

LF Representation

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Construct terms by applying generators:

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1 : nat = succ zero  
2 : nat = succ 1
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Correctness of Representation

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Why is this a correct representation?

Correctness of Representation

Representation is **adequate** iff isomorphic to informal syntax

In this case, we define a bijection between:

- numbers n **nat**
- LF canonical forms (synonym: “terms”) $M : \text{nat}$

$$\begin{aligned}\lceil \text{zero} \rceil &= \text{zero} \\ \lceil \text{succ}(n) \rceil &= \text{succ} \lceil n \rceil\end{aligned}$$

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Injectivity: easy

Surjectivity: take our word for it for now

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The **addition judgement** $m + n$ is p states that the sum of m and n is p .

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Addition Judgement

$$\frac{}{\text{zero} + n \text{ is } n} \qquad \frac{m + n \text{ is } p}{\text{succ}(m) + n \text{ is } \text{succ}(p)}$$

Example: $1 + 1 \text{ is } 2$

$$\frac{\frac{}{\text{zero} + \text{succ}(\text{zero}) \text{ is } \text{succ}(\text{zero})}}{\text{succ}(\text{zero}) + \text{succ}(\text{zero}) \text{ is } \text{succ}(\text{succ}(\text{zero}))}}$$

Judgements as Types

The judgement is represented by a **family of types** in LF:

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`add : nat -> nat -> nat -> type.`

Adequacy: The LF type `add` $\ulcorner m \urcorner \ulcorner n \urcorner \ulcorner p \urcorner$ classifies **derivations** of *$m + n$ is p* .

∇ derives *$m + n$ is p*

iff

$\ulcorner \nabla \urcorner : \text{add} \ulcorner m \urcorner \ulcorner n \urcorner \ulcorner p \urcorner$

Rules as Generators

$$\frac{}{\text{zero} + n \text{ is } n}$$

$$\frac{m + n \text{ is } p}{\text{succ}(m) + n \text{ is } \text{succ}(p)}$$

Each addition rule is represented by a **generator**:

add/z : add zero N N.

add/s : add (succ M) N (succ P)
 <- add M N P.

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add/z : add zero N N.

add/s : add (succ M) N (succ P)
 <- add M N P.

Use capital letters for **schema variables**

Representing Syntax and Judgements

Let's type these rules into Twelf!

Exercise

Define multiplication:

$$\frac{}{\text{zero} * n \text{ is zero}}$$

$$\frac{m * n \text{ is } p \quad n + p \text{ is } p'}{\text{succ}(m) * n \text{ is } p'}$$

Syntax for multiple premises:

```
rule : conclusion
      <- premise1
      <- premise2.
```

Basic Twelf Skills

- 1 Representing syntax and judgements
- 2 **Totality of judgements**
- 3 Proving metatheorems

Totality of Judgements

We can use Twelf to **verify** totality of judgements:

For all m **nat** and n **nat**,
there exists a p **nat** such that $m + n$ is p

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- addition is total, with first two args as inputs; third as output
- addition is total with **mode** + + -

Totality of Judgements

Recast as a statement about LF terms:

For all m **nat** and n **nat**,
there exists a p **nat** such that $m + n$ is p

becomes

For all $m:\text{nat}$ and $n:\text{nat}$,
there exists some $p:\text{nat}$ and $\nabla:\text{add } m \ n \ p$

Totality of Judgements

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- 3 **Termination**: no circular definitions.

Syntax:

- **%mode**: state the mode
- **%total**: coverage and termination

Totality of Judgements

`add : nat -> nat -> nat -> type.`

Thm: $\forall m:\text{nat}$ and $n:\text{nat}$, $\exists p:\text{nat}$ and $\nabla : \text{add } m \ n \ p$

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Twelf declarations:

```
%mode add +M +N -P.
```

```
%total M (add M _ _).
```


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The first two args are inputs and the third is an output

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Asks Twelf to prove totality by induction on the first argument

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Thm: $\forall m:\text{nat}$ and $n:\text{nat}$, $\exists p:\text{nat}$ and $\nabla:\text{add } m \ n \ p$

Twelf declarations:

```
%mode add +M +N -P.  
%worlds () (add _ _ _).  
%total M (add M _ _).
```

add is to be proved total on **closed** terms of type nat

Totality of Judgements

Let's go over these declarations in more detail in Twelf...

Exercise

Can you use `add` to do subtraction? Is it total...

- with mode `+ - + ?`
- with mode `- + + ?`

Can you use `mult` to do (exact) division? Is it total...

- with mode `+ - + ?`
- with mode `- + + ?`

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Metatheory With Twelf

Much standard meta-theory is **easily** mechanized using Twelf.

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We will consider several examples today:

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We will consider several examples today:

- Warm-up: arithmetic

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- Type safety for a small language (but scales to serious languages such as Standard ML).

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We will consider several examples today:

- Warm-up: arithmetic
- Type safety for a small language (but scales to serious languages such as Standard ML).
- More advanced examples later

Example

$$\frac{}{\text{zero} + n \text{ is } n} \text{ add/z} \qquad \frac{m + n \text{ is } p}{\text{succ}(m) + n \text{ is } \text{succ}(p)} \text{ add/s}$$

Lemma: For all m **nat**, we can derive $m + \text{zero} \text{ is } m$.

Proof: induction on m .

Example

Lemma: For all $m \text{ nat}$, we can derive $m + \text{zero}$ is m .

Case for $m = \text{zero}$:

To show: $\text{zero} + \text{zero}$ is zero .

True by rule:

$$\frac{}{\text{zero} + \text{zero} \text{ is } \text{zero}} \text{ add/z}$$

Example

Lemma: For all $m \text{ nat}$, we can derive $m + \text{zero}$ is m .

Case for $m = \text{succ}(m')$:

To show: $\text{succ}(m') + \text{zero}$ is $\text{succ}(m')$

By IH we get derivation ∇ of $m' + \text{zero}$ is m' .

By rule:

$$\frac{\nabla \quad m' + \text{zero} \text{ is } m'}{\text{succ}(m') + \text{zero} \text{ is } \text{succ}(m')} \text{ add/s}$$

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Metatheory With Twelf

Lemma: For all $m \text{ nat}$, we can derive $m + \text{zero is } m$.

- Proof defines a **transformation** that generates a derivation of $m + \text{zero is } m$ for every number m .
- Can represent transformation as a binary relation that relates each number to a derivation
- Show **totality** of relation to prove the theorem.

Sound familiar?

Metatheory With Twelf

We already know how to define total relations! E.g.

```
add : nat -> nat -> nat -> type.  
%mode add +M +N -P.  
%total M (add M _ _).
```


Metatheory With Twelf

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add : nat -> nat -> nat -> type.  
%mode add +M +N -P.  
%total M (add M - _).
```

What is a Twelf proof?

- Define a relation as an LF type family
- Get Twelf to prove it `%total`

Representing Theorem Statements

Lemma: For all $m \text{ nat}$, we can derive $m + \text{zero}$ is m .

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Lemma: For all $M:\text{nat}$, there exists a $D:(\text{add } M \text{ zero } M)$.

Theorem statement becomes type family, mode, worlds:

```
rhzero : {m : nat} add m zero m -> type.  
%mode rhzero +M -D.  
%worlds () (rhzero _ _).
```

Representing Proofs

Lemma: For all $M:\text{nat}$, there exists a $D:(\text{add } M \text{ zero } M)$.

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rhzero : {m : nat} add m zero m -> type.  
%mode rhzero +M -D.  
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```

Proof becomes:

- Cases = generators populating the family
- Twelf checks %total:

Lemma: For all $M:\text{nat}$, there exists a $D:(\text{add } M \text{ zero } M)$
and $D':(\text{rhzero } M \ D)$.

First Case

Lemma: For all $m \text{ nat}$, we can derive $m + \text{zero}$ is m .

Case for $m = \text{zero}$:

To show: $\text{zero} + \text{zero}$ is zero .

True by rule:

$$\frac{}{\text{zero} + \text{zero} \text{ is } \text{zero}} \text{ add/z}$$

Second Case

Lemma: For all $m \text{ nat}$, we can derive $m + \text{zero} \text{ is } m$.

Case for $m = \text{succ}(m')$:

To show: $\text{succ}(m') + \text{zero} \text{ is } \text{succ}(m')$

By IH we get derivation ∇ of $m' + \text{zero} \text{ is } m'$.

By rule:

$$\frac{\nabla \quad m' + \text{zero} \text{ is } m'}{\text{succ}(m') + \text{zero} \text{ is } \text{succ}(m')} \text{ add/s}$$

Metatheory in Twelf

We can use this methodology to prove $\forall\exists$ -type properties of representations:

$$\forall M_1 : A_1 \dots \forall M_k : A_k \exists N_1 : B_1 \dots \exists N_l : B_l \top$$

This is sufficient for a **large body** of metareasoning!

Another example

Analogous lemma for successor'ing the right-hand arg:

Lemma: *For all nats m, n, p ,
if $m + n$ is p then $m + \text{succ}(n)$ is $\text{succ}(p)$.*

Proof: induction over derivation of $m + n$ is p .

Exercise

Commutativity: *For all nats m, n, p ,
if $m + n$ is p then $n + m$ is p .*

Hint: do induction over the derivation of $m + n$ is p and use previous two lemmas!

Recap and Prospectus

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Next: apply these tools to study a simple programming language.

Part II

Representing Variable Binding

Syntax for MinML

$\tau ::= \text{num} \mid \tau_1 \Rightarrow \tau_2$

$e ::= z \mid s(e)$ (constructors for *num*)
| $ifz(e, e_0, x.e_1)$ (case-analysis for *num*)
| $fn\ x:\tau.e \mid e_1\ e_2$ (functions and application)
| x (variables)

Write τ **type** and e **exp** for syntax generated by these grammars.

Representing Syntax

Remember the methodology:

- Syntactic category becomes an LF type
- Each rule becomes a generator in the LF signature.

Types are easy:

$$\tau ::= \textit{num} \mid \tau_1 \Rightarrow \tau_2$$

Let's represent this in Twelf...

Representing Syntax

Encoding function:

$$\begin{aligned}\lceil \mathit{num} \rceil &= \mathit{num} \\ \lceil \tau_1 \Rightarrow \tau_2 \rceil &= \mathit{arr} \lceil \tau_1 \rceil \lceil \tau_2 \rceil\end{aligned}$$

Adequacy: $\lceil \tau \rceil$ is a bijection between τ **type** and LF terms of type τ

Representing Syntax

Expressions:

$$e ::= z \mid s(e) \mid e_1 e_2 \quad (\text{easy})$$
$$\mid \text{ifz}(e, e_0, x.e_1) \mid \text{fn } x:\tau.e \quad (\text{binding constructs})$$
$$\mid x \quad (\text{variables})$$

Let's do the easy ones first...

Representing Syntax

Encoding function:

$$\begin{aligned}\lceil z \rceil &= z \\ \lceil s(e) \rceil &= s \lceil e \rceil \\ \lceil e_1 e_2 \rceil &= \text{app } \lceil e_1 \rceil \lceil e_2 \rceil \\ \lceil fn x:\tau.e \rceil &= ??? \\ \lceil ifz(e, e_0, x.e_1) \rceil &= ??? \\ \lceil x \rceil &= ???\end{aligned}$$

How do we represent binding constructs and variables?

Representing Bindings

$fn\ x:\tau.e$ constructs an expression out of two things:

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- May be **renamed**, preserving pronoun structure:

$fn\ x:num.s(x)$ is $fn\ y:num.s(y)$.

Representing Bindings

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- domain type τ
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- May be **renamed**, preserving pronoun structure:

$$fn\ x:num.s(x) \text{ is } fn\ y:num.s(y).$$

- May be **substituted** by an expression, preserving pronoun structure:

$$[s(s(z))/x](s(x)) \text{ is } s(s(s(z)))$$

Representing Bindings

Notate the formation of fn with a **general** judgement:

$$\frac{\tau \text{ type} \quad x \text{ exp} \mid e \text{ exp}}{fn \ x:\tau.e \text{ exp}}$$

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- The variable x may occur within e .

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Notate the formation of fn with a **general** judgement:

$$\frac{\tau \text{ type} \quad x \text{ exp} \mid e \text{ exp}}{fn \ x:\tau.e \ \text{exp}}$$

- The variable x may occur within e .
- General judgement can be α -converted

$$(x \ \text{exp} \mid e \ \text{exp}) \equiv_{\alpha} (y \ \text{exp} \mid [y/x]e \ \text{exp})$$

Representing Bindings

Notate the formation of fn with a **general** judgement:

$$\frac{\tau \text{ type} \quad x \text{ exp} \mid e \text{ exp}}{fn \ x:\tau.e \ \text{exp}}$$

- The variable x may occur within e .
- General judgement can be α -converted

$$(x \ \text{exp} \mid e \ \text{exp}) \equiv_{\alpha} (y \ \text{exp} \mid [y/x]e \ \text{exp})$$

- Substitution is valid: $[e/x]e_2 \ \text{exp}$ whenever $e \ \text{exp}$.

Higher-Order Abstract Syntax

$$\frac{\tau \text{ type} \quad x \text{ exp} \mid e \text{ exp}}{fn \ x:\tau.e \ \text{exp}}$$

represented by LF generator:

`fn : tp -> (exp -> exp) -> exp.`

Higher-Order Abstract Syntax

$$\frac{\tau \text{ type} \quad x \text{ exp} \mid e \text{ exp}}{fn \ x:\tau.e \ \text{exp}}$$

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Uses **higher-order functions** to express binding and scope!

Higher-Order Abstract Syntax

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represented by LF generator:

$$fn : \tau p \rightarrow (\text{exp} \rightarrow \text{exp}) \rightarrow \text{exp}.$$

Uses higher-order functions to express binding and scope!

Expression with a free variable $x \text{ exp} \mid e \text{ exp}$
represented by
LF function of type $\text{exp} \rightarrow \text{exp}$

Higher-Order Abstract Syntax

Expression with a free variable x **exp** | e **exp**
represented by
LF function of type **exp** \rightarrow **exp**

LF functions **exp** \rightarrow **exp**:

- Intro: ($[x:\text{exp}] M$) , where $M:\text{exp}$ assuming $x:\text{exp}$

Higher-Order Abstract Syntax

Expression with a free variable x $\text{exp} \mid e \text{exp}$
represented by
LF function of type $\text{exp} \rightarrow \text{exp}$

LF functions $\text{exp} \rightarrow \text{exp}$:

- Intro: $([x:\text{exp}] M)$, where $M:\text{exp}$ assuming $x:\text{exp}$
- Elim: $M1 M2$ (like we've been writing all along)

Higher-Order Abstract Syntax

$\text{fn} : \text{tp} \rightarrow (\text{exp} \rightarrow \text{exp}) \rightarrow \text{exp}.$

Representation:

$$\begin{aligned} \lceil \text{fn } x:\tau.e \rceil &= \text{fn } \lceil \tau \rceil ([x:\text{exp}] \lceil e \rceil) \\ \lceil x \rceil &= x \end{aligned}$$

- LF function represents the binding and scope of x in e
- object-language variables represented by LF variables

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Representation:

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- LF function represents the binding and scope of x in e
- object-language variables represented by **LF variables**

Exercise

$$\lceil \text{ifz}(e, e_0, x.e_1) \rceil = ???$$

Adequacy

Basic idea: Define a bijection between e **exp** and $M : \text{exp}$

Questions:

- How does the bijection treat free variables?

Adequacy

Basic idea: Define a bijection between e **exp** and $M : \text{exp}$

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- What is an **isomorphism** of syntax with binding?

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- Do all LF functions of type $\text{exp} \rightarrow \text{exp}$ represent syntax?

Part III

LF and Adequacy

LF and Adequacy

- ① Simply-typed LF: Canonical forms; substitution
- ② Adequacy of `exp`
- ③ Dependent types

LF and Adequacy

- ① Simply-typed LF: Canonical forms; substitution
- ② Adequacy of `exp`
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The LF Type Theory

LF is a dependently typed λ -calculus

Simply typed fragment:

Types $A ::= a \mid A1 \rightarrow A2$

Canonical forms $M ::= x \ M1 \dots Mn \mid c \ M1 \dots Mn \mid [x] \ M$

The LF Type Theory

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Types $A ::= a \mid A1 \rightarrow A2$

Canonical forms $M ::= x \ M1 \dots Mn \mid c \ M1 \dots Mn \mid [x] \ M$

Base types like `nat`, `tp`, `exp`

The LF Type Theory

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Simply typed fragment:

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Canonical forms $M ::= x \ M1 \dots Mn \mid c \ M1 \dots Mn \mid [x] \ M$

Function types like $\text{exp} \rightarrow \text{exp}$

The LF Type Theory

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Simply typed fragment:

Types $A ::= a \mid A1 \rightarrow A2$

Canonical forms $M ::= x \ M1 \dots Mn \mid c \ M1 \dots Mn \mid [x] \ M$

Variable or constant applied to canonical forms (no β -redices)

The LF Type Theory

LF is a dependently typed λ -calculus

Simply typed fragment:

Types $A ::= a \mid A1 \rightarrow A2$

Canonical forms $M ::= x \ M1 \dots Mn \mid c \ M1 \dots Mn \mid [x] \ M$

λ -abstraction

Canonical Forms

Types $A ::= a \mid A1 \rightarrow A2$
Canonical forms $M ::= x \ M1 \dots Mn \mid c \ M1 \dots Mn \mid [x] \ M$

Signature $\Sigma ::= \varepsilon \mid \Sigma, a:\text{type} \mid \Sigma, c:A$

Context $\Gamma ::= \varepsilon \mid \Gamma, x:A$

Canonical Forms

Types $A ::= a \mid A1 \rightarrow A2$
Canonical forms $M ::= x \ M1 \dots Mn \mid c \ M1 \dots Mn \mid [x] \ M$

Signature $\Sigma ::= \varepsilon \mid \Sigma, a:\text{type} \mid \Sigma, c:A$

Context $\Gamma ::= \varepsilon \mid \Gamma, x:A$

Base types (`nat`) and constants (`zero`, `succ`) declared in signature

Canonical Forms

Types $A ::= a \mid A1 \rightarrow A2$

Canonical forms $M ::= x \ M1 \dots Mn \mid c \ M1 \dots Mn \mid [x] \ M$

Signature $\Sigma ::= \varepsilon \mid \Sigma, a:\text{type} \mid \Sigma, c:A$

Context $\Gamma ::= \varepsilon \mid \Gamma, x:A$

Variables bound in context

Canonical Forms

<i>Types</i>	$A ::= a \mid A1 \rightarrow A2$
<i>Canonical forms</i>	$M ::= x \mid M1 \dots Mn \mid c \mid M1 \dots Mn \mid [x] \mid M$
<i>Signature</i>	$\Sigma ::= \varepsilon \mid \Sigma, a : \text{type} \mid \Sigma, c : A$
<i>Context</i>	$\Gamma ::= \varepsilon \mid \Gamma, x : A$

Inductive definition of canonical forms:

$$\Gamma \vdash_{\Sigma} M : A$$

M is a canonical form of type A in signature Σ and context Γ

Canonical Forms

Constants:

$$\frac{c : (A_1 \rightarrow \dots \rightarrow A_n \rightarrow a) \in \Sigma \quad \Gamma \vdash_{\Sigma} M_i : A_i}{\Gamma \vdash_{\Sigma} c \ M_1 \dots M_n : a}$$

Canonical Forms

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Example:

$$\frac{\text{arr} : (\text{tp} \rightarrow \text{tp} \rightarrow \text{tp}) \in \Sigma \quad \frac{\vdots}{\varepsilon \vdash_{\Sigma} \text{num} : \text{tp}} \quad \frac{\vdots}{\varepsilon \vdash_{\Sigma} \text{num} : \text{tp}}}{\varepsilon \vdash_{\Sigma} (\text{arr} \ \text{num} \ \text{num}) : \text{tp}}$$

Canonical Forms

Variables:

$$\frac{x : (A_1 \rightarrow \dots \rightarrow A_n \rightarrow a) \in \Gamma \quad \Gamma \vdash_{\Sigma} M_i : A_i}{\Gamma \vdash_{\Sigma} x M_1 \dots M_n : a}$$

Canonical Forms

Variables:

$$\frac{x : (A_1 \rightarrow \dots \rightarrow A_n \rightarrow a) \in \Gamma \quad \Gamma \vdash_{\Sigma} M_i : A_i}{\Gamma \vdash_{\Sigma} x M_1 \dots M_n : a}$$

Example:

$$\frac{}{x : \text{exp} \vdash_{\Sigma} x : \text{exp}}$$

Canonical Forms

Functions:

$$\frac{\Gamma, x:A \vdash_{\Sigma} M : B}{\Gamma \vdash_{\Sigma} ([x] M) : A \rightarrow B}$$

Canonical Forms

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Example:

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Canonical Forms

Functions:

$$\frac{\Gamma, x:A \vdash_{\Sigma} M : B}{\Gamma \vdash_{\Sigma} ([x] M) : A \rightarrow B}$$

Example:

$$\frac{\Gamma, x:\text{exp} \vdash_{\Sigma} x : \text{exp}}{\Gamma \vdash_{\Sigma} ([x] x) : \text{exp} \rightarrow \text{exp}}$$

This is the only rule for $A \rightarrow B$, so canonical forms are η -expanded

Induction on Canonical Forms

Canonical forms $M ::= x \ M_1 \dots M_n \mid c \ M_1 \dots M_n \mid [x] \ M$

Adequacy and metatheory are based on structural induction on canonical forms

Example: What are the canonical forms of type `exp`?

Canonical Forms of exp

What are the canonical forms of type exp
in context $x_1:\text{exp}, \dots, x_n:\text{exp}$?

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- `fn M1 M2` where `M1:tp` and `M2:(exp -> exp)` are canon.

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`fn M1 ([x] M)` where `M` is canonical assuming `x:exp`

No “exotic terms”!

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- z
- $s \ M$ where M is canonical at exp
- $\text{ifz } M \ M0 \ M1$ where $M, M0$ canonical at exp
and $M1$ canonical at $(\text{exp} \rightarrow \text{exp})$

No “exotic terms”!

LF Substitution

$[M' / x] M$
where $\Gamma, x:A \vdash_{\Sigma} M : B$ and $\Gamma \vdash_{\Sigma} M' : A$

LF Substitution

$$[M' / x] M$$

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But canonical forms are **not** closed under substitution:

$$\begin{aligned} A &= \text{exp} \rightarrow \text{exp} \\ M' &= [x] x \\ M &= x N \end{aligned}$$

Substitution results in β -redex $([x] x) N$!

LF Substitution

$[M' / x] M$
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But canonical forms are **not** closed under substitution

Two solutions:

LF Substitution

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Two solutions:

- Old: Allow non-canonical forms, use $\beta\eta$ -equality

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Two solutions:

- Old: Allow non-canonical forms, use $\beta\eta$ -equality
- New: Use **hereditary substitution** to directly compute the canonical result of substitution

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But canonical forms are **not** closed under substitution

Two solutions:

- Old: Allow non-canonical forms, use $\beta\eta$ -equality
- New: Use **hereditary substitution** to directly compute the canonical result of substitution

Twelf: allows non-canonical forms in source text;
uses hereditary substitution under the hood

LF and Adequacy

- ① Simply-typed LF: Canonical forms; substitution
- ② Adequacy of exp
- ③ Dependent types

Adequacy

Basic idea: Define a bijection between e **exp** and $M : \text{exp}$

Questions:

- How does the bijection treat free variables?
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Adequacy for expressions with binding

Isomorphism specified by

- A context-indexed family of bijections

$$\begin{array}{c} x_1 \mathbf{exp}, \dots, x_n \mathbf{exp} \mid e \mathbf{exp} \\ \longleftrightarrow \\ x_1:\mathbf{exp}, \dots, x_n:\mathbf{exp} \vdash_{\Sigma_{\mathbf{exp}}} \ulcorner e \urcorner : \mathbf{exp}. \end{array}$$

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- That is **compositional**: respects substitution

$$\ulcorner [e/x]e' \urcorner = \ulcorner [e \urcorner/x] \urcorner e' \urcorner$$

Encoding of Syntax

Define a family of bijections

$$\begin{array}{c} x_1 \mathbf{exp}, \dots, x_n \mathbf{exp} \mid e \mathbf{exp} \\ \longleftrightarrow \\ x_1:\mathbf{exp}, \dots, x_n:\mathbf{exp} \vdash_{\Sigma_{\mathbf{exp}}} \ulcorner e \urcorner : \mathbf{exp}. \end{array}$$

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Variables:

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Encoding of Syntax

Application:

$$\begin{array}{c} x_1 \mathbf{exp}, \dots, x_n \mathbf{exp} \mid e_1 e_2 \mathbf{exp} \\ \longleftrightarrow \\ x_1:\mathbf{exp}, \dots, x_n:\mathbf{exp} \vdash_{\Sigma_{\mathbf{exp}}} \mathbf{app} \ulcorner e_1 \urcorner \ulcorner e_2 \urcorner \end{array}$$

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Encoding of Syntax

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Encoding of Syntax

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Encoding of Syntax

Functions:

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Compositionality

Demand that encoding commutes with substitution:

if $x \mathbf{exp} \mid e' \mathbf{exp}$ and $e \mathbf{exp}$, then $\ulcorner [e/x]e' \urcorner = \ulcorner e \urcorner / x \urcorner \ulcorner e' \urcorner$.

where

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Compositionality

$$\ulcorner [e/x]e' \urcorner = \ulcorner e \urcorner / x \urcorner \ulcorner e' \urcorner.$$

Proves that object language's notion of substitution is faithfully represented by LF substitution

Proving Adequacy

$$\begin{aligned} x_1 \mathbf{exp}, \dots, x_n \mathbf{exp} \mid e \mathbf{exp} \\ \longleftrightarrow \\ x_1:\mathbf{exp}, \dots, x_n:\mathbf{exp} \vdash_{\Sigma_{\mathbf{exp}}} \lceil e \rceil : \mathbf{exp}. \end{aligned}$$

- $\lceil e \rceil$ is a function Proof: induction on e

Proving Adequacy

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- $\lceil e \rceil$ is a function Proof: induction on e
- Compositionality Proof: induction on e

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- $\lceil e \rceil$ is a function Proof: induction on e
- Compositionality Proof: induction on e
- $\lceil e \rceil$ is injective Proof: induction on e

Proving Adequacy

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- $\lceil e \rceil$ is a function Proof: induction on e
- Compositionality Proof: induction on e
- $\lceil e \rceil$ is injective Proof: induction on e
- $\lceil e \rceil$ is surjective Proof: induction on canonical forms

$$x_1:\mathbf{exp}, \dots, x_n:\mathbf{exp} \vdash_{\Sigma_{\mathbf{exp}}} M : \mathbf{exp}.$$

Canonical Forms of exp

What are the canonical forms of type exp
in context $x_1:\text{exp}, \dots, x_n:\text{exp}$?

- x_i
- $\text{app } M1 \ M2$ where $M1$ and $M2$ are canonical at exp
- $\text{fn } M1 \ M2$ where $M1:\text{tp}$ and $M2:(\text{exp} \rightarrow \text{exp})$ are canon.
 $\text{fn } M1 \ ([x] \ M)$ where M is canonical assuming $x:\text{exp}$
- z
- $s \ M$ where M is canonical at exp
- $\text{ifz } M \ M0 \ M1$ where $M, M0$ canonical at exp
and $M1$ canonical at $(\text{exp} \rightarrow \text{exp})$

Encoding of Syntax

Suppose we allow

$f : (\text{exp} \rightarrow \text{exp}), x_1 : \text{exp}, \dots, x_n : \text{exp}$

Surjectivity fails: **new** canonical forms

- $f \text{ zero}$

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- $\text{ifz } M \ M0 \ ([x] \ f \ x)$

Lesson: adequacy depends crucially on the **world** (set of LF contexts)

Adequacy

Basic idea: Define a bijection between e **exp** and $M : \text{exp}$

Questions:

- How does the bijection treat free variables?
- What is an isomorphism of syntax with binding?
- Do all LF functions of type $\text{exp} \rightarrow \text{exp}$ represent syntax?

No “exotic terms”

Suppose context has the form $x_1:\text{exp}, \dots, x_n:\text{exp}$

- By canonical forms, every term $\text{exp} \rightarrow \text{exp}$ is $([x:\text{exp}] M)$

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- By canonical forms, every term $\text{exp} \rightarrow \text{exp}$ is $([x:\text{exp}] M)$
- M has type

$$x_1:\text{exp}, \dots, x_n:\text{exp}, \mathbf{x:\text{exp}} \vdash_{\Sigma_{\text{exp}}} M:\text{exp}$$

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- Every such M represents an expression with free vars $x_1 \dots x_n, x$

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- So $([x] M)$ represents an expression with a bound variable

Intuitive explanation: exp is a base type, so the only thing you can do with $x:\text{exp}$ is use it (e.g., no case analysis)

LF and Adequacy

- ① Simply-typed LF: Canonical forms; substitution
- ② Adequacy of `exp`
- ③ **Dependent types**

The LF Type Theory

Dependently-typed LF:

Canonical forms $M ::= x \ M_1 \dots M_n \mid c \ M_1 \dots M_n \mid [x] \ M$

Types $A ::= a \mid A_1 \rightarrow A_2$

Kinds $K ::= \text{type} \mid \{x:A\} \ K$

The LF Type Theory

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Kinds $K ::= \text{type} \mid \{x:A\} \ K$

Family constants applied to args: `add M N P`

The LF Type Theory

Dependently-typed LF:

Canonical forms $M ::= x \ M_1 \dots M_n \mid c \ M_1 \dots M_n \mid [x] \ M$

Types $A ::= a \ M_1 \dots M_n \mid \{x:A\} \ B$

Kinds $K ::= \text{type} \mid \{x:A\} \ K$

Dependent function types:

$\text{add}/z : \{n:\text{nat}\} \ \text{add} \ \text{zero} \ n \ n$

The LF Type Theory

Dependently-typed LF:

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Kinds $K ::= \text{type} \mid \{x:A\} \ K$

Kinds of family constants:

$\text{rhzero} : \{n:\text{nat}\} \ \{-:\text{add } n \ \text{zero } n\} \ \text{type}$

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Kinds of family constants:

$\text{rhzero} : \{n:\text{nat}\} \ \text{add} \ n \ \text{zero} \ n \ \rightarrow \ \text{type}$

($A \ \rightarrow \ B$ means $\{-:A\} \ B$)

Dependent Application

$$\frac{\Gamma \vdash_{\Sigma} M1 : \{x:A\} B \quad \Gamma \vdash_{\Sigma} M2 : A}{\Gamma \vdash_{\Sigma} (M1 \ M2) : [M2/x]B}$$

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Example:

`add/z : {n:nat} add zero n n`

`0+2=2 : add zero 2 2 = add/z 2`

Dependent Application

$$\frac{\Gamma \vdash_{\Sigma} M1 : \{x:A\} B \quad \Gamma \vdash_{\Sigma} M2 : A}{\Gamma \vdash_{\Sigma} (M1 \ M2) : [M2/x]B}$$

Example:

```
add/s : {m:nat} {n:nat} {p:nat}
        add m n p
        -> add (succ m) n (succ p)
partial : add 0 1 2 -> add 1 1 3
         = [d] add/s 0 1 2 d
```

Summary

- $\Gamma \vdash_{\Sigma} M : A$

Inductive definition of **canonical forms** M of type A ,
in context Γ and signature Σ

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Inductive definition of **canonical forms** M of type A ,
in context Γ and signature Σ

- Adequacy: compositional bijection between
object-language syntax
and
canonical forms of a specified type in specified contexts

$$\begin{array}{c} x_1 \mathbf{exp}, \dots, x_n \mathbf{exp} \mid e \mathbf{exp} \\ \longleftrightarrow \\ x_1:\mathbf{exp}, \dots, x_n:\mathbf{exp} \vdash_{\Sigma_{\mathbf{exp}}} \ulcorner e \urcorner : \mathbf{exp}. \end{array}$$

Consequences of Adequacy

An adequate representation **obviates** the object language itself!

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Experience has shown that it **improves our understanding** of an object language to formalize it in LF.

Metatheory can be mechanized as proofs by **induction on canonical forms**.

Induction on Canonical Forms

```
add : nat -> nat -> nat -> type
%total M (add M _ _ _).
```

Proof: induction on canonical forms of type `nat`
(= structural induction on syntax)

Induction on Canonical Forms

```
rhsucc : add M N P -> add M (succ N) (succ P) -> type.  
%total D (rhsucc D _).
```

Proof: induction on canonical forms of type `add M N P`
(= rule induction on derivations)

Part IV

Representing Hypothetical Judgements

Judgements

Static Semantics:

- $\Gamma \vdash e : \tau$

Dynamic Semantics:

- e **val**
- $e \mapsto e'$

Static semantics

Static semantics for **natural numbers**

$$\frac{}{\Gamma \vdash z : num} \text{ of/z} \quad \frac{\Gamma \vdash e : num}{\Gamma \vdash s(e) : num} \text{ of/s}$$

$$\frac{\Gamma \vdash e : num \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x : num \vdash e_1 : \tau}{\Gamma \vdash ifz(e, e_0, x.e_1) : \tau} \text{ of/ifz}$$

Static semantics for **functions and application**

$$\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ of/app} \quad \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash fn x:\tau_1.e : \tau_1 \rightarrow \tau_2} \text{ of/fn}$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} dx$$

Higher-Order Rules

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \mathit{fn} \ x : \tau_1 . e : \tau_1 \Rightarrow \tau_2}$$

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$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \mathit{fn} \ x : \tau_1 . e : \tau_1 \Rightarrow \tau_2} \quad \text{i.e.} \quad \frac{x \ \mathbf{exp} \mid x : \tau_1 \vdash e : \tau_2}{\mathit{fn} \ x : \tau_1 . e : \tau_1 \Rightarrow \tau_2}$$

Really a **general, hypothetical judgement**:

Higher-Order Rules

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \mathit{fn} \ x : \tau_1 . e : \tau_1 \Rightarrow \tau_2} \quad \text{i.e.} \quad \frac{x \ \mathbf{exp} \mid x : \tau_1 \vdash e : \tau_2}{\mathit{fn} \ x : \tau_1 . e : \tau_1 \Rightarrow \tau_2}$$

Really a **general, hypothetical judgement**:

- General in a **fresh parameter** x

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Really a **general, hypothetical judgement**:

- General in a fresh parameter x
- Hypothetical in a **new axiom** stating that x has type τ_1

Higher-Order Metavariables

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \mathit{fn} \ x:\tau_1.e : \tau_1 \Rightarrow \tau_2} \quad \text{i.e.} \quad \frac{x \mathbf{exp} \mid x : \tau_1 \vdash e : \tau_2}{\mathit{fn} \ x:\tau_1.e : \tau_1 \Rightarrow \tau_2}$$

- e is a metavar standing for *terms that may mention x*

Higher-Order Metavariables

$$\frac{\Gamma, x : \tau_1 \vdash e_x : \tau_2}{\Gamma \vdash \mathit{fn} \ x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2} \quad \text{i.e.} \quad \frac{x \mathbf{exp} \mid x : \tau_1 \vdash e_x : \tau_2}{\mathit{fn} \ x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2}$$

- e is a metavar standing for *terms that may mention x*
- Can make this explicit by writing e_x

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- e is a metavar standing for *terms that may mention x*
- Can make this explicit by writing e_x
- Look at e_- in isolation as a **higher-order metavariable**

Hypothetical Judgements in LF

$$\Gamma \vdash e : \tau$$

Idea: just as with syntax, use **LF variables** to represent hypotheses

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- Binary relation on e and τ , where Γ is represented by the LF context

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$$\Gamma \vdash e : \tau$$

Idea: just as with syntax, use **LF variables** to represent hypotheses

- Typing judgement is **not** a three-place relation on Γ , e , and τ
- Binary relation on e and τ , where Γ is represented by the LF context

of : exp -> tp -> type.

Higher-Order Rules in LF

$$\frac{\Gamma, x : \tau_1 \vdash e_x : \tau_2}{\Gamma \vdash \mathit{fn} \ x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2} \quad \text{i.e.} \quad \frac{x \ \mathbf{exp} \mid x : \tau_1 \vdash e_x : \tau_2}{\mathit{fn} \ x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2}$$

Higher-order rules are represented using **higher-order types**

```
of/fn : of (fn T1 ([x] E x)) (arr T1 T2)
        <- ({x:exp} of x T1 -> of (E x) T2).
```

Higher-Order Rules in LF

$$\frac{\Gamma, x : \tau_1 \vdash e_x : \tau_2}{\Gamma \vdash \mathit{fn} \ x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2} \quad \text{i.e.} \quad \frac{x \ \mathbf{exp} \mid x : \tau_1 \vdash e_x : \tau_2}{\mathit{fn} \ x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2}$$

represented by

of/fn : of (fn T1 ([x] E x)) (arr **T1** T2)
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- T1:tp and T2:tp correspond to τ_1 and τ_2

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 <- ({x:exp} of x T1 -> of (E x) T2).

- T1:tp and T2:tp correspond to τ_1 and τ_2
- E:(exp -> exp) corresponds to higher-order metavar e_{-}

Higher-Order Rules in LF

$$\frac{\Gamma, x : \tau_1 \vdash e_x : \tau_2}{\Gamma \vdash \mathit{fn} \ x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2} \quad \text{i.e.} \quad \frac{x \ \mathbf{exp} \mid x : \tau_1 \vdash e_x : \tau_2}{\mathit{fn} \ x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2}$$

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General, hypothetical judgement: body is typed relative to

Higher-Order Rules in LF

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of/fn : of (fn T1 ([x] E x)) (arr T1 T2)
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General, hypothetical judgement: body is typed relative to

- A **fresh variable** x

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$$\frac{\Gamma, x : \tau_1 \vdash e_x : \tau_2}{\Gamma \vdash \text{fn } x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2} \quad \text{i.e.} \quad \frac{x \text{ **exp** } \mid x : \tau_1 \vdash e_x : \tau_2}{\text{fn } x:\tau_1.e_x : \tau_1 \Rightarrow \tau_2}$$

represented by

of/fn : of (fn T1 ([x] E x)) (arr T1 T2)
 <- ({x:exp} of x T1 -> of (E x) T2).

General, hypothetical judgement: body is typed relative to

- A fresh variable x
- A **new axiom** stating that x has type $E1$

Static semantics

Static semantics for **natural numbers**

$$\frac{}{\Gamma \vdash z : \text{num}} \text{ of/z} \quad \frac{\Gamma \vdash e : \text{num}}{\Gamma \vdash s(e) : \text{num}} \text{ of/s}$$

$$\frac{\Gamma \vdash e : \text{num} \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x : \text{num} \vdash e_1 : \tau}{\Gamma \vdash \text{ifz}(e, e_0, x.e_1) : \tau} \text{ of/ifz}$$

Static semantics for **functions and application**

$$\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ of/app} \quad \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \text{fn } x : \tau_1 . e : \tau_1 \rightarrow \tau_2} \text{ of/fn}$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \text{ dx}$$

Adequacy and Worlds

Typing introduces **parameters** and **hypotheses**

Consider a **world** (set of contexts) consisting of **blocks** of the form

$$x : \text{exp}, dx : \text{of } x \ T \quad (\text{for some } T : \text{tp})$$

Adequacy relative to hypotheses in this world:

$$\begin{array}{c} \nabla \text{ derives } x_1 : \tau_1, \dots \vdash e : \tau \\ \text{iff} \\ x_1 : \text{exp}, dx_1 : \text{of } x_1 \ulcorner \tau_1 \urcorner, \dots \vdash \ulcorner \nabla \urcorner : \text{of } \ulcorner e \urcorner \ulcorner \tau \urcorner \end{array}$$

Adequacy and Worlds

Worlds are declared in Twelf using `%block` and `%worlds`:

```
%block of_block
      :                block {x:exp} {dx:of x T}.
%worlds (of_block) (of _ _).
```

Twelf checks: all assumptions made in rules are have specified form

Adequacy and Worlds

Worlds are declared in Twelf using `%block` and `%worlds`:

```
%block of_block
      : some {T:tp} block {x:exp} {dx:of x T}.
%worlds (of_block) (of _ _).
```

Twelf checks: all assumptions made in rules are have specified form

Dynamic semantics

Dynamic semantics for **functions and application**

$$\frac{}{fn\ x:\tau.e\ \mathbf{val}} \textit{value/fn}$$

$$\frac{e_1 \mapsto e'_1}{e_1\ e_2 \mapsto e'_1\ e_2} \textit{step/app/fn} \qquad \frac{e_1\ \mathbf{val} \quad e_2 \mapsto e'_2}{e_1\ e_2 \mapsto e_1\ e'_2} \textit{step/app/arg}$$

$$\frac{e_2\ \mathbf{val}}{(fn\ x:\tau.e)\ e_2 \mapsto [e_2/x]e} \textit{step/app/beta}$$

Dynamic semantics

Dynamic semantics for **natural numbers**

$$\frac{}{z \text{ val}} \text{ value/z} \qquad \frac{e \text{ val}}{s(e) \text{ val}} \text{ value/s}$$

$$\frac{e \mapsto e'}{s(e) \mapsto s(e')} \text{ step/s} \qquad \frac{e \mapsto e'}{\text{ifz}(e, e_0, x.e_1) \mapsto \text{ifz}(e', e_0, x.e_1)} \text{ step/ifz/arg}$$

$$\frac{}{\text{ifz}(z, e_0, x.e_1) \mapsto e_0} \text{ step/ifz/z}$$

$$\frac{e \text{ val}}{\text{ifz}(s(e), e_0, x.e_1) \mapsto [e/x]e_1} \text{ step/ifz/s}$$

Part V

Type Safety for MinML

Type Safety

Progress: If $\cdot \vdash e : \tau$, then e **val** or else $e \mapsto e'$ (for some e').

Preservation: If $\cdot \vdash e : \tau$ and $e \mapsto e'$ then $\cdot \vdash e' : \tau$

Progress theorem

Progress: If $\cdot \vdash e : \tau$, then e **val** or else $e \mapsto e'$ (for some e').

Proof: induction on derivation of $\cdot \vdash e : \tau$.

Progress theorem (zero)

If $\cdot \vdash e : \tau$, then e **val** or else $e \mapsto e'$ (for some e').

Case: $\overline{z : num}$ *of/z*

($e = z, \tau = num$)

- By rule *value/z*, z **val**

Progress theorem (fn)

If $\cdot \vdash e : \tau$, then e **val** or else $e \mapsto e'$ (for some e').

Case: $\frac{x : \tau_1 \vdash e_0 : \tau_2}{fn\ x:\tau.e_0 : \tau_1 \rightarrow \tau_2}$ *of/fn*

($e = fn\ x:\tau.e_0, \tau = \tau_1 \rightarrow \tau_2$)

- By rule *value/fn*, $fn\ x:\tau.e_0$ **val**

Progress theorem (app)

If $\cdot \vdash e : \tau$, then e **val** or else $e \mapsto e'$ (for some e').

Case: $\frac{e_1 : \tau' \rightarrow \tau \quad e_2 : \tau'}{e_1 e_2 : \tau}$ *of/app*

($e = e_1 e_2$)

- By the induction hypothesis, e_1 **val** or else $e_1 \mapsto e'_1$, and additionally e_2 **val** or else $e_2 \mapsto e'_2$
- Case analysis to prove that $e_1 e_2 \mapsto e'$:
 - If $e_1 \mapsto e'_1$, then $e_1 e_2 \mapsto e'_1 e_2$ by rule *step/app/fn*
 - If e_1 **val** and $e_2 \mapsto e'_2$, then $e_1 e_2 \mapsto e_1 e'_2$ by rule *step/app/arg*
 - If e_1 **val** and e_2 **val**, then **by canonical forms**, $e_1 = \text{fn } x:\tau'.e_0$ and so $(\text{fn } x:\tau'.e_0) e_2 \mapsto [e_2/x]e_0$ by rule *step/app/beta*

Progress theorem (app)

If $\cdot \vdash e : \tau$, then e **val** or else $e \mapsto e'$ (for some e').

Case:
$$\frac{e_1 : \tau' \rightarrow \tau \quad e_2 : \tau'}{e_1 e_2 : \tau} \text{ of/app}$$

($e = e_1 e_2$)

- By the induction hypothesis, e_1 **val** or else $e_1 \mapsto e'_1$, and additionally e_2 **val** or else $e_2 \mapsto e'_2$
- Case analysis to prove that $e_1 e_2 \mapsto e'$:
 - If $e_1 \mapsto e'_1$, then $e_1 e_2 \mapsto e'_1 e_2$ by rule *step/app/fn*
 - If e_1 **val** and $e_2 \mapsto e'_2$, then $e_1 e_2 \mapsto e_1 e'_2$ by rule *step/app/arg*
 - If e_1 **val** and e_2 **val**, then **by further case analysis on the derivation of $e_1 : \tau \rightarrow \tau'$** , $e_1 = \text{fn } x:\tau'.e_0$ and so $(\text{fn } x:\tau'.e_0) e_2 \mapsto [e_2/x]e_0$ by rule *step/app/beta*

Progress theorem (successor)

If $\cdot \vdash e : \tau$, then e **val** or else $e \mapsto e'$ (for some e').

Case: $\frac{e_1 : num}{s(e_1) : num}$ *of/s*

$(e = s(e_1), \tau = num)$

- By the induction hypothesis, e_1 **val** or else $e_1 \mapsto e'_1$
- Case analysis to prove $s(e_1)$ **val** or else $s(e_1) \mapsto e'$:
 - If e_1 **val**, then $s(e_1)$ **val** by rule *value/s*
 - If $e_1 \mapsto e'_1$ for some e'_1 , then $s(e_1) \mapsto s(e'_1)$ by rule *step/s*

Progress theorem (ifz)

If $\cdot \vdash e : \tau$, then e **val** or else $e \mapsto e'$ (for some e').

Case:
$$\frac{e_{arg} : num \quad e_0 : \tau \quad x : num \vdash e_1 : \tau}{ifz(e_{arg}, e_0, x.e_1) : \tau} \text{ of/ifz}$$

($e = ifz(e_{arg}, e_0, x.e_1)$)

- By the induction hypothesis, e_{arg} **val** or else $e_{arg} \mapsto e'_{arg}$
- Case analysis to prove that $ifz(e_{arg}, e_0, x.e_1) \mapsto e'$:
 - If $e_{arg} \mapsto e'_{arg}$, then $ifz(e_{arg}, e_0, x.e_1) \mapsto ifz(e'_{arg}, e_0, x.e_1)$ by rule *step/ifz/arg*.
 - If e_{arg} **val**, then by **canonical forms**, either
 - $e_{arg} = z$, and $ifz(e_{arg}, e_0, x.e_1) \mapsto e_0$ by rule *step/ifz/z*.
 - $e_{arg} = s(e_{pred})$ where e_{pred} **val**, and $ifz(e_{arg}, e_0, x.e_1) \mapsto e_0$ by rule *step/ifz/s*.

Part VI

Bonus Slides: Canonical LF

Canonical LF

Formation judgements of LF:

$$\Gamma \vdash_{\Sigma} K \text{ kind}$$

$$\Gamma \vdash_{\Sigma} A \Rightarrow K$$

$$\Gamma \vdash_{\Sigma} M \Leftarrow A \quad \Gamma \vdash_{\Sigma} R \Rightarrow A$$

$$\vdash_{\Sigma} \Gamma \text{ ok} \quad \vdash \Sigma \text{ ok}$$

Canonical objects are **analyzed**, atomic objects are **synthesized**.

Canonical LF

Substitution judgements of LF:

$$[M/x]K = K'$$

$$[M/x]A = A'$$

$$[M/x]N = N' \quad [M/x]R = M'$$

Substitution judgements of LF:

$$[M/x]K = K'$$

$$[M/x]A = A'$$

$$[M/x]N = N' \quad [M/x]R = M'$$

The **critical case** threatens termination:

$$[\lambda_{y:A} M/x](x N) = [N/y]M$$

Substitution judgements of LF:

$$[M/x]K = K'$$

$$[M/x]A = A'$$

$$[M/x]N = N' \quad [M/x]R = M'$$

The **critical case** threatens termination:

$$[\lambda_{y:A} M/x](x N) = [N/y]M$$

But the **erased type** (dependency-free simple type) of the substituting object gets smaller!

Atomic Objects

Variables and constants:

$$\overline{\Gamma \vdash_{\Sigma_1, c:A, \Sigma_2} c \Rightarrow A} \quad \overline{\Gamma_1, x:A, \Gamma_2 \vdash_{\Sigma} x \Rightarrow A}$$

Atomic Objects

Variables and constants:

$$\overline{\Gamma \vdash_{\Sigma_1, c:A, \Sigma_2} c \Rightarrow A} \quad \overline{\Gamma_1, x:A, \Gamma_2 \vdash_{\Sigma} x \Rightarrow A}$$

Function application:

$$\frac{\Gamma \vdash_{\Sigma} R \Rightarrow \Pi_{x:A_1} A_2 \quad \Gamma \vdash_{\Sigma} M \Leftarrow A_1 \quad [M/x]A_2 = A}{\Gamma \vdash_{\Sigma} R M \Rightarrow A}$$

Canonical Objects

Atomic objects of base type are canonical:

$$\frac{\Gamma \vdash_{\Sigma} R \Rightarrow A \quad A \neq \Pi_{x:A_1} A_2}{\Gamma \vdash_{\Sigma} R \Leftarrow A}$$

Canonical Objects

Atomic objects of base type are canonical:

$$\frac{\Gamma \vdash_{\Sigma} R \Rightarrow A \quad A \neq \prod_{x:A_1} A_2}{\Gamma \vdash_{\Sigma} R \Leftarrow A}$$

Abstractions are canonical at higher type:

$$\frac{\Gamma, x : A_1 \vdash_{\Sigma} M \Leftarrow A_2}{\Gamma \vdash_{\Sigma} \lambda_{x:A_1} M_2 \Leftarrow \prod_{x:A_1} A_2}$$

Type Families

Constants:

$$\overline{\Gamma \vdash_{\Sigma_1, a:K, \Sigma_2} a \Rightarrow K}$$

Type Families

Constants:

$$\overline{\Gamma \vdash_{\Sigma_1, a:K, \Sigma_2} a \Rightarrow K}$$

Family instantiation:

$$\frac{\Gamma \vdash_{\Sigma} A \Rightarrow \Pi_{x:A_1} K_2 \quad \Gamma \vdash_{\Sigma} M \Leftarrow A_1 \quad [M/x]K_2 = K}{\Gamma \vdash_{\Sigma} AM \Rightarrow K}$$

Type Families

Products of families:

$$\frac{\Gamma \vdash_{\Sigma} A_1 \Rightarrow \text{type} \quad \Gamma, x : A_1 \vdash_{\Sigma} A_2 \Rightarrow \text{type}}{\Gamma \vdash_{\Sigma} \prod_{x:A_1} A_2 \Rightarrow \text{type}}$$

The kind of types:

$$\overline{\Gamma \vdash_{\Sigma} \text{type kind}}$$

The kind of types:

$$\overline{\Gamma \vdash_{\Sigma} \text{type kind}}$$

Product of a kind family:

$$\frac{\Gamma \vdash_{\Sigma} A_1 \Rightarrow \text{type} \quad \Gamma, x : A_1 \vdash_{\Sigma} K_2 \text{ kind}}{\Gamma \vdash_{\Sigma} \prod_{x:A_1} K_2 \text{ kind}}$$